Global bifurcation of planar and spatial periodic solutions from the polygonal relative equilibria for the n-body problem

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Abstract

Given n point masses turning in a plane at a constant speed, this paper deals with the global bifurcation of periodic solutions for the masses, in that plane and in space. As a special case, one has a complete study of n identical masses on a regular polygon and a central mass. The symmetries of the problem are used in order to find the irreducible representations, the linearization, and with the help of the orthogonal degree theory, all the symmetries of the bifurcating branches.

Keywords: ring configuration, $(n+1)$-body problem, global bifurcation, equivariant degree theory.

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1 Introduction

Consider n point masses turning at a constant speed in a plane around some fixed point. A relative equilibrium of this configuration is a stationary solution of the equations in the rotating coordinates. In this paper we analyze the bifurcation of periodic solutions from a general relative equilibrium. As a special case, we give a complete study of the polygonal relative equilibrium where there are n identical masses arranged on a regular polygon and a central mass. This model was posed by Maxwell in order to explain the stability of Saturn's rings. For this polygonal equilibrium, we give a full analysis for the bifurcation of planar and spatial periodic solutions. We will prove that, according to the value of the central mass, there are up to $2n$ branches of planar periodic solutions, with different symmetries, and up to n additional branches which are, if some non resonance condition is met, in the space $\mathbb{R}^3$ with non trivial vertical components.

More precisely, in rotating coordinates and after a scaling in time, we look for 2π-periodic solutions $x_j(t)$, for $j = 0, 1, ..., n$, where $j = 0$ corresponds to the central body of mass μ while each element of the ring has mass 1, and frequency ν. We shall prove (see the exact theorems and graphs in the core of the paper) that, for each integer $k = 1, ..., n$, there are one or two intervals (one unbounded) of values of the central mass with global branches of periodic solutions bifurcating from the relative equilibrium. In fact, for $k = 2, ..., n-2$ and for $k = 1, n-1$ and $n > 6$, there is an unbounded interval of values of μ with two branches of planar periodic solutions (short and long period) with the symmetries described below. For $k = 1, n-1$ and $n = 3, 4, 5, 6$ there is only one bifurcating branch in the unbounded interval, while for $n = 2$ there is one branch for any value of μ.

For $k = 2, ..., n-2$, there is a bounded interval where one has one bifurcating global branch for $\mu < \mu_k$, where μ_k is the value of the central mass where one has the global branch of relative equilibria obtained in [19]. For $k = 2, ..., n/2$, there are two bifurcating branches for $\mu_k < \mu < m_0$.

For $k = n$, and any $n > 1$, there is a global branch of planar solutions starting at $\nu = 1$.

Finally, for $k = 1, ..., n$ and any μ, there is a global branch of solutions with a non trivial vertical coordinate, which is odd in time while the horizontal coordinates are π-periodic, with the possible exception of a finite number of resonances at a finite number of values of μ's.

With respect to the symmetries, the solutions in the rotating system, with the horizontal coordinates $u_j(t)$ and vertical coordinate $z_j(t)$, satisfy the following properties, with $\zeta = 2\pi/n$: $u_0(t) = 0$, if k and n have some common factor, and if not, $u_0(t+\zeta) = e^{-ik'\zeta}u_0(t)$. where k' is such that $kk' = 1$, modulo n.

Furthemore, for $j = 1, ..., n$:

$$u_j(t) = e^{ij\zeta}u_n(t + kj\zeta) \text{ and } z_j(t) = z_n(t + kj\zeta).$$

Except for a possible finite number of μ's and ν's, for μ positive, bounded and different from μ_k, due to resonances, these solutions are geometrically different.

By global branch, we mean that there is a continuum of solutions starting at the ring configuration and a specific value of the frequency, which goes to infinity in the norm of the solution or in the period $1/\nu$, or goes to collision, or otherwise goes to other relative equilibria in such a way that the sum of the jumps in the orthogonal degrees is zero.

There is a vast literature on the n-body problem, many on the bifurcation of relative equilibria and on their stability, fewer on periodic solutions. We shall mention only the papers which are closer to our research, that is [26], [32], [27], [33], among others. The case of a satellite attracted by an array of n masses was treated in [18], and previously, for the case of $n = 2$, by A. Maciejewski and S. Rybicki in [25], using the orthogonal degree for the action of $SO(2)$. See also [17] for details.

For our particular setting, the linearization of the system at a critical point is a $2n \times 2n$ matrix, which is non invertible, due to the rotational symmetry.

These facts imply that the study of the spectrum of the linearization is not an easy task and that the classical bifurcation results for periodic solutions may not be applied directly. However, we shall use the change of variables proved in our previous paper, [19], in order to give not only this spectrum but also the consequences for the symmetries of the solutions.

The present paper is the last part of our application of the orthogonal degree to similar dynamical problems. In [19], we had a complete study of the stationary problem, while in [20] we have considered the analitically simpler case of vortices and quasi-paralel filaments. Since this paper is a continuation of [19], we shall use the results in that paper, but we shall recall all the important notions.

The next section is devoted to the mathematical setting of the problem, with the symmetries involved in the general case and in the particular situation of the regular polygon. Then, we give, in the following three sections, the preliminary results needed in order to apply the orthogonal degree theory developed in [24], that is the global Liapunov-Schmidt reduction, the study of the irreducible representations, for the general and the polygonal situations, with the change of variables of [19], and the symmetries associated to these representations. In the next section, we prove our bifurcation results and, in the following section, we give the analysis of the spectrum in the general and the polygonal situations, with the complete results on the type of solutions which bifurcate from the relative equilibrium, in the plane and also in space. In the final section, we give some remarks on the resonance condition, on the stability and on the case of charged particles.

Remark 1 *Due to the rotational symmetry, the trivial equilibrium of the ring comes with an $SO(2)$- orbit and the problem is degenerate, with at least a one-dimensional kernel. In many previous papers with degeneracies, the authors look for solutions in fixed-point subspaces (odd maps, fixing the phase as in the classical Hopf bifurcation), in order to be able to apply simpler methods.*

In rotating coordinates, the full group is a semidirect product of T^2 with $\mathbb{Z}_2$ for the general problem, and the additional group of permutations S_n, for the polygonal relative equilibrium.

The action of T^2 is given by time shifts and rotations in the plane, the action of the element $\kappa_1 \in \mathbb{Z}_2$ is given by $\kappa_1 x_j(t) = diag(1,-1,1)x_j(-t)$. Thus, the group contains two different subgroups isomorphic to $O(2)$. Moreover, if one fixes the bifurcation map by the action of κ_1, then the equilibrium must be a collinear central configuration, which is not the case of the polygonal equilibrium, or of the more general setting.

However, the polygonal equilibrium is fixed by the action

$$\tilde{\kappa} x_j(t) = diag(1,-1,1)x_{n-j}(-t),$$

which is a coupling of κ_1 with the permutation of the bodies in D_n. If one restricts the problem to the fixed-point subspace of $\tilde{\kappa}$, then one proves bifurcation of periodic solutions only for the blocks given by $k = n$ and $k = n/2$.

The solutions obtained for the restricted map are the following:

- *For $k = n$ and planar solutions,the bifurcation branch consists of solutions with $u_0(t) = 0$ and $u_j(t) = e^{ij\zeta}u_n(t)$, that is all the elements move as they were in the ring configuration. This branch was constructed in an explicit way in [28], by reducing the problem to a 6-dimensional dynamical system and a normal form argument.*

- *For $k = n$ and spatial solutions, the branch is made of solutions where the ring moves as a whole and the central body makes the contrary movement in order to stabilize the forces, that is $z_j(t) = z_n(t)$ and $z_0 = -\sum z_j$. This solution is called an oscillating ring in [28].*

- *For $k = n/2$ and planar solutions, the bifurcation has the central body fixed at the center, and two polygons of $n/2$ bodies pulsing each one as a whole.*

- *For $k = n/2$ and spatial solutions, one gets the well known Hip-Hop orbits. This kind of solutions appears first in the paper [12] without the central body. Later on, in [28] for a big central body in order to explain the pulsation of the Saturn ring, where they are called kink solutions. Finally, there is a proof in [2] when there is no central body.*

For the other k's, the linearization on the fixed-point subspace of $\tilde{\kappa}$ is a complex matrix with non-negative determinant as a real matrix. This fact implies that one may not use a classical degree argument or other simple analytical proofs. See the remarks in the 4th section for a more detailed discussion.

Note that if one wishes to restrict the problem to one of the isotropy subspaces described in this paper, one has to use anyway the orthogonal degree. Our approach enables us to treat the complete problem in one single system.

Note also that one cannot ask for an arrangement of the bodies such that the central body remains on the x-axis and the bodies in the ring are symmetrical with respect to that axis, for all times, since the dynamical equations are not satisfied; this is why $\tilde{\kappa}$ has to reverse time.

Remark 2 *As pointed out above, the n-body problem has been the object of many papers, with different techniques and different purposes. For few bodies one may use a normal form approach, which gives good local information but is difficult to apply to large systems as the one we have here. The same thing applies to the Liapunov center theorem or the decomposition with a central variety. For the case of relative equilibria, one may consult [15] and [7], among others.*

Besides the orthogonal degree, which is designed for equivariant problems with large symmetries, one may try to use other topological equivariant tools such as the equivariant Conley index or equivariant variational indices which effectively give a Weinstein-Moser theorem, [3], but from a finite orbit to a T^n-orbit, which is not the case here.

Finally, variational techniques have been quite successful in treating the problem of existence of special solutions such as the choreographies and the hip-hop solutions. In particular, [16], [13] and [14], classify all the possible groups which

give solutions which are minimizers of the action without collisions, in the plane for the first paper and in space for the other two. Thus, the issue is different from ours, since one has the proof of the existence of a solution in the large, with a specific symmetry, but no multiplicity results or localization of the solutions.

For the choreographies, following the seminal paper [9], with no central mass, one has studies with more that 3 bodies and minima of the action in [5] and [4], for instance.

In the case of hip-hop solutions, the first papers such as [12] and [28], as well as [2], did not use variational methods. On the other hand these methods were successful in [10] and [34].

One would like to relate all the different solutions obtained by all these methods. In the case of the analytical local solutions the linearization is also used in the application of any degree theory, although in the later case one may treat degenerate problems or large kernels. The relation between topological solutions and variational solutions is not easy to establish. For instance, it is one of the goal of [8] to prove that the bifurcation branch of hip-hop solutions may connect to variational solutions in the large. In particular, they give a study of the vertical bifurcation for the cases $k = n$ and $k = n/2$, with no central mass and some conditions of non-resonance. The fact that there is no proof of the uniqueness of the minimizers implies that there is no rigourous proof of a global continuation and a connection of the hip-hop solutions to the eight choreography.

One of the main advantages of the orthogonal degree is that it applies to problems which are not necessarily variational, but present conserved quantities. Furthermore, deformations are easy to construct. From the theoretical point of view the theory has to be extended to the action of non-abelian groups and to abstract infinite dimensional spaces (in the case of hamiltonian systems of first order the degrees don't stabilize completely: see [24], Lemma 3.6, p.264, and one has to use a simple reduction to a finite number of modes, but without loss of information: see the section on the Liapunov-Schmidt reduction). See also[31] for the case of gradients. Furthermore, the orthogonal degree gives global branches of solutions, it takes into account all symmetries and proves the existence of many solutions, even in the case of possible resonances, where the issue is to know if the obtained solution is not a subharmonic (see the last section or [29]). As pointed out above, these solutions are different.

2 Setting the problem

Let $q_j(t) \in \mathbb{R}^3$ be the position of the body $j \in \{1, ..., n\}$ with mass m_j. Define the 3×3 matrices $\bar{I} = diag(1, 1, 0)$ and $\bar{J} = diag(J, 0)$, where J is the standard symplectic matrix in $\mathbb{R}^2$. In rotating coordinates $q_j(t) = e^{\sqrt{\omega} t \bar{J}} u_j(t)$, Newton equations of the n bodies are

$$m_j \ddot{u}_j + 2m_j \sqrt{\omega} \bar{J} \dot{u}_j = \omega m_j \bar{I} u_j - \sum_{i=1 (i \neq j)}^{n} m_i m_j \frac{u_j - u_i}{\|u_j - u_i\|^{\alpha+1}}.$$

Define the vector u as $(u_1, ..., u_n)$, the matrix of masses $\mathcal{M}$ as $diag(m_1 I, ..., m_n I)$ and the matrix $\bar{\mathcal{J}}$ as $diag(\bar{J}, ..., \bar{J})$. Then, Newton equations of the n bodies in vector form are

$$\mathcal{M}\ddot{u} + 2\sqrt{\omega}\mathcal{M}\bar{\mathcal{J}}\dot{u} = \nabla V(u) \text{ with} \tag{1}$$
$$V(u) = \frac{\omega}{2}\sum_{j=1}^{n} m_j \left\|\bar{I}u_j\right\|^2 + \sum_{i<j} m_i m_j \phi_\alpha(\|u_j - u_i\|).$$

The function $\phi_\alpha(x)$ is defined such that $\phi'_\alpha(x) = -1/x^\alpha$. The gravitational force is the particular case $\phi_2(x) = 1/x$.

Critical points of the potential V correspond to relative equilibria of the n-body problem. This was part of the study done in [19], Proposition 1.

Remark 3 *If one replaces, in the change of coordinates, the term $e^{\sqrt{\omega}t\bar{J}}$ with a complex factor $\varphi(t)$ (taking q_j and u_j as complex functions instead of a planar vector), where φ satifies the equation*

$$\varphi'' = -\omega\varphi/|\varphi|^3,$$

the equations become

$$(\mathcal{M}\ddot{u} + 2(\varphi)'/(\varphi)\mathcal{M}\dot{u})|\varphi|^3 = \nabla V(u).$$

In particular, the stationary solutions of this system are the same solutions studied in [19], and the solutions for φ are those of the Kepler problem, where ω is the central mass. Thus, the solutions for q_j are now ellipses, parabolas or hyperbolas, instead of circular orbits. One may have also total collapse or solutions like $ct^{2/3}$, with $|c| = (9\omega/2)^{1/3}$. The bifurcation analysis for periodic solutions starting near the relative equilibria may be performed in this slightly more general setting.

To prove bifurcation of periodic solutions near a relative equilibrium, we need to change variables as $x(t) = u(t/\nu)$. In this way, the $2\pi/\nu$-periodic solutions of the differential equation are 2π-periodic solutions of the bifurcation operator

$$f : H^2_{2\pi}(\mathbb{R}^{3n}\backslash\Psi) \times \mathbb{R}^+ \to L^2_{2\pi} \tag{2}$$
$$f(x,\nu) = -\nu^2\mathcal{M}\ddot{x} - 2\sqrt{\omega}\nu\bar{\mathcal{J}}\mathcal{M}\dot{x} + \nabla V(x).$$

The set $\Psi = \{x \in \mathbb{R}^{3n} : x_i = x_j\}$ is the collision set, when two or more of the bodies collide, and $H^2_{2\pi}(\mathbb{R}^{3n}\backslash\Psi)$ is the open subset, consisting of the collision-free periodic (and continuous) functions, of the Sobolev space $H^2(\mathbb{R}^{3n})$,

$$H^2_{2\pi}(\mathbb{R}^{3n}\backslash\Psi) = \{x \in H^2_{2\pi}(\mathbb{R}^{3n}) : x_i(t) \neq x_j(t)\}.$$

Definition 4 *We define the action of $\Gamma = \mathbb{Z}_2 \times SO(2)$ in $\mathbb{R}^{3n}$ as*

$$\rho(\kappa) = \mathcal{R} \text{ and } \rho(\theta) = e^{-\bar{\mathcal{J}}\theta},$$

where $R = diag(1,1,-1)$ and $\mathcal{R} = diag(R, ..., R)$, where κ and θ are elements of Γ.

The group $\mathbb{Z}_2$ acts by reflection on the z-axis and the group $SO(2)$ acts by rotation in the (x, y)-plane. Clearly, the potential V is invariant by the action of the group Γ. Consequently, the gradient ∇V is Γ-orthogonal. This means that ∇V is Γ-equivariant, $\rho(\gamma)\nabla V(x) = \nabla V(\rho(\gamma)x)$, and that $\nabla V(x)$ is orthogonal to the infinitesimal generator,

$$A_1 x = \frac{\partial}{\partial\theta}|_{\theta=0} e^{-\bar{\mathcal{J}}\theta} x = -\bar{\mathcal{J}} x.$$

These facts can be proved directly from the definitions.

Given that ∇V is Γ-equivariant and the equation is autonomous, then the map f is $\Gamma \times S^1$-equivariant, where the action of S^1 is by time translation. Moreover, the infinitesimal generator of the action of S^1 in time is $A_0 x = \dot{x}$. Hence, the map f is $\Gamma \times S^1$-orthogonal because of the equalities

$$\langle f(x), \dot{x}\rangle_{L^2_{2\pi}} = -\frac{\nu^2}{2} \left\| \mathcal{M}^{1/2}\dot{x} \right\|^2 |_0^{2\pi} - 2\nu\sqrt{\omega} \left\langle \bar{\mathcal{J}}\mathcal{M}^{1/2}\dot{x}, \mathcal{M}^{1/2}\dot{x} \right\rangle_{L^2_{2\pi}} + V(x)|_0^{2\pi} = 0,$$

$$\left\langle f(x), \bar{\mathcal{J}}x \right\rangle_{L^2_{2\pi}} = \nu^2 \left\langle \mathcal{M}^{1/2}\dot{x}, \bar{\mathcal{J}}\mathcal{M}^{1/2}\dot{x} \right\rangle_{L^2_{2\pi}} - \nu\sqrt{\omega} \left\| \mathcal{M}^{1/2}x \right\|^2 |_0^{2\pi} + \int_0^{2\pi} \left\langle \nabla V, \bar{\mathcal{J}}x \right\rangle = 0.$$

Remark 5 *It is well known that the Newton equations, in fixed coordinates, are invariant under the action of the group of symmetries $\mathbb{R}^3 \rtimes O(3)$ which gives the conservation laws of linear and angular momenta. In rotating coordinates, the potential V is invariant under the action of the subgroup $\mathbb{Z}_2 \times \mathbb{R} \times O(2)$.*

Since the matrices $\bar{\mathcal{J}}$ and $\mathcal{R}$ anticommute, $\mathcal{R}\bar{\mathcal{J}} = -\bar{\mathcal{J}}\mathcal{R}$, then the operator $f(x)$ is equivariant under the action of the full group, described by

$$(T^2 \cup \kappa_1 T^2) \times (\mathbb{R} \cup \kappa\mathbb{R}). \tag{3}$$

The actions of $(\theta, \varphi) \in T^2$ and κ_1 are given by $(\theta, \varphi)x = e^{-\bar{\mathcal{J}}\theta} x(t + \varphi)$ and $\kappa_1 x(t) = \mathcal{R}_1 x(-t)$, where $\mathcal{R}_1 = diag(R_1, ..., R_1)$ with $R_1 = diag(1, -1, 1)$. The action of $\mathbb{R} \cup \kappa\mathbb{R}$ is given by the spatial reflection κ, which was already defined, and the action of $r \in \mathbb{R}$ is a translation on the z-axis, $rx = x + re$ with $e_3 = (0, 0, 1)$ and

$$e = (e_3, ..., e_3).$$

The action of the group $\mathbb{R}$ gives a conservation law of the linear momentum with respect to the spatial axis. Consequently, any spatial translation of a relative equilibrium is also an equilibrium. Since the orthogonal degree is defined only for compact abelian groups, [24], p. 70, then we have to restrict the full group of symmetries (3) to the abelian subgroup $\Gamma \times S^1 = T^2 \times \langle\kappa\rangle$. However, in order to follow with this procedure we have to deal with the missing symmetries of $\mathbb{R}$ by hand. The easy way to do it consists in restricting the map f to the orthogonal space to $e \in \mathbb{R}^{3n}$.

Definition 6 *We define the map $f : \mathcal{W} \cap H^2_{2\pi} \to \mathcal{W}$ as the restriction of the bifurcation map $f(x)$ to the space*

$$\mathcal{W} = \{x \in L^2_{2\pi}(\mathbb{R}^{3n} \backslash \Psi) : \int_0^{2\pi} (x \cdot e) dt = 0\}. \tag{4}$$

From the previous remark, the potential V is invariant under the action of $\mathbb{R}$. In our approach, this means that $\nabla V(x)$ is orthogonal to the infinitesimal generator of the action $\nabla V(x) \cdot e = 0$, a fact which may be proved directly from the definitions. Consequently, the map $f : \mathcal{W} \cap H^2_{2\pi} \to \mathcal{W}$ is well defined since

$$\int_0^{2\pi} (f(x) \cdot e) dt = \int_0^{2\pi} (\nabla V(x) \cdot e) dt = 0.$$

2.1 General relative equilibria

All relative equilibria are planar. Moreover, any spatial translation of a relative equilibrium is also an equilibrium. Nevertheless, the only relative equilibrium in $\mathcal{W}$ of the family of translations is the one in the (x, y)-plane. Therefore, the positions $(u_j, 0)$ form a relative equilibrium in $\mathcal{W}$ if they satisfy the relations

$$\omega u_i = \sum_{j=1\ (j \neq i)}^{n} m_j \frac{u_i - u_j}{\|u_i - u_j\|^{\alpha+1}} \text{ with } u_j \in \mathbb{R}^2.$$

Remark 7 *Since the potential is homogenous, then any scaling of a relative equilibrium is also an equilibrium. Hence, in principle we may choose the relative equilibrium with $\omega = 1$. However, we shall leave ω as a parameter because it is easier to fix the norm of the polygonal equilibrium than the frequency.*

Since every relative equilibrium x_0 is planar, then the action of $\kappa \in \Gamma$ leaves fixed x_0. Therefore, the isotropy subgroup of x_0 in $\Gamma \times S^1$ is $\Gamma_{x_0} \times S^1$ with $\Gamma_{x_0} = \langle \kappa \rangle = \mathbb{Z}_2$. Thus, the orbit of x_0 is isomorphic to the group S^1: the orbit of the equilibrium x_0 consists of all the rotations of x_0 in the (x, y)-plane.

Notice that the generator of the spatial group Γ at x_0 is $A_1 x_0 = -\bar{\mathcal{J}} x_0$, then $-\bar{\mathcal{J}} x_0$ is tangent to the orbit and must be in the kernel of $D^2 V(x_0)$.

Now, since $\mathbb{Z}_2$ is in the isotropy subgroup of x_0, then $D^2 V(x_0)$ is $\mathbb{Z}_2$-equivariant. From Schur's lemma, see [24], p. 18, the matrix $D^2 V(x_0)$ must have a block diagonal form. In the following proposition we prove directly this fact.

Proposition 8 *Let $\mathcal{A}_{ij}$ be the 3×3 blocks such that*

$$D^2 V(x_0) = (\mathcal{A}_{ij})_{ij=1}^n,$$

then the matrices $\mathcal{A}_{ij}$ have the diagonal form

$$\mathcal{A}_{ij} = diag(A_{ij}, a_{ij}).$$

Let d_{ij} be the distance between u_i and u_j, and let (x_j, y_j) be the components of u_j. For $i \neq j$, we have that $a_{ij} = m_i m_j / d_{ij}^{\alpha+1}$ and

$$A_{ij} = -\frac{m_i m_j}{d_{ij}^{\alpha+3}} \begin{pmatrix} (\alpha+1)(x_i - x_j)^2 - d_{ij}^2 & (\alpha+1)(x_i - x_j)(y_i - y_j) \\ (\alpha+1)(x_i - x_j)(y_i - y_j) & (\alpha+1)(y_i - y_j)^2 - d_{ij}^2 \end{pmatrix}. \quad (5)$$

Moreover, for $i = j$ we have the equalities

$$a_{ii} = -\sum_{j=1\ (j\neq i)}^{n} a_{ij} \text{ and } A_{ii} = \omega m_i I - \sum_{j\neq i} A_{ij}.$$

Proof. The function $\phi_\alpha(d_{ij})$ has the second derivative matrix

$$D^2_{u_i}\phi_\alpha(d_{ij}) = \frac{\alpha+1}{d_{ij}^{\alpha+3}} \begin{pmatrix} (x_i-x_j)^2 & (x_i-x_j)(y_i-y_j) & 0 \\ (x_i-x_j)(y_i-y_j) & (y_i-y_j)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{I}{d_{ij}^{\alpha+1}}.$$

For $i \neq j$, since $\nabla_{u_i}\phi(d_{ij}) = -\nabla_{u_j}\phi(d_{ij})$, then

$$\mathcal{A}_{ij} = m_i m_j D_{u_j}\nabla_{u_i}\phi(d_{ij}) = -m_i m_j D^2_{u_i}\phi(d_{ij}) = diag(A_{ij}, a_{ij}).$$

And for $i = j$ the matrix $\mathcal{A}_{ii}$ satisfies

$$\mathcal{A}_{ii} = \omega m_i \bar{I} + \sum_{j\neq i} m_i m_j D^2_{u_i}\phi(d_{ij}) = \omega m_i \bar{I} - \sum_{j\neq i} \mathcal{A}_{ij} = diag(A_{ii}, a_{ii}).$$

■

If we analyze the n-body problem in the plane instead of the space, then the potential of the planar problem has the Hessian $D^2V(x_0) = (A_{ij})_{ij=1}^n$. Because of this fact, we say that $(A_{ij})_{ij=1}^n$ will give the planar spectrum and $(a_{ij})_{ij=1}^n$ the spatial spectrum.

2.2 The polygonal equilibrium

Identifying the real and complex planes, a relative equilibrium is given by the positions $(a_j, 0) \in \mathbb{R}^3$ with $a_j \in \mathbb{C}$. Defining $\zeta = 2\pi/n$, the polygonal equilibrium is formed by $n+1$ bodies: one body, with mass $m_0 = \mu$, at $a_0 = 0$, and one body for each $j \in \{1, ..., n\}$, with mass $m_j = 1$, at $a_j = e^{ij\zeta}$.

It is easy to prove, for instance [19], Proposition 1, that the ring configuration $\bar{a} = (a_0, ..., a_n)$ is a relative equilibrium when $\omega = \mu + s_1$, with

$$s_1 = \frac{1}{2^\alpha}\sum_{j=1}^{n-1} \frac{1}{\sin^{(\alpha-1)}(j\zeta/2)}.$$

In this paper, the norm of the polygonal equilibrium is fixed and $\omega = \mu + s_1$ is a free parameter. In the paper [19], Theorem 24, we have proved bifurcation of relative equilibria from the polygonal equilibrium with the parameter μ . Now, we wish to analyze the bifurcation of planar and spatial periodic solutions.

Definition 9 *Let S_n be the group of permutations of $\{1, ..., n\}$. We define the action of S_n in $\mathbb{R}^{2(n+1)}$ as*

$$\rho(\gamma)(x_0, x_1, ..., x_n) = (x_0, x_{\gamma(1)}, ..., x_{\gamma(n)}).$$

As for the general case, the map f is $(\mathbb{Z}_2\times SO(2))\times S^1$-orthogonal. Moreover, since n of the bodies have equal mass, then the gradient ∇V is S_n-equivariant. Therefore, we may consider the map f as $\Gamma\times S^1$-orthogonal with respect to the abelian group

$$\Gamma=\mathbb{Z}_2\times\mathbb{Z}_n\times SO(2),$$

where $\mathbb{Z}_n$ is the subgroup of S_n generated by $\zeta(j)=j+1$.

Let $\tilde{\mathbb{Z}}_n$ be the subgroup of Γ generated by $(\zeta,\zeta)\in\mathbb{Z}_n\times SO(2)$ with $\zeta=2\pi/n\in SO(2)$. Since the actions of (ζ,ζ), given by taking the j-th element in the plane x_j to $e^{-J\zeta}x_{j+1}$, for $j\in 1,.,.,.,n$ and x_0 to $e^{-J\zeta}x_0$, and $\kappa\in\mathbb{Z}_2$, fixing the plane, leave the equilibrium $\bar{a}$ fixed, then the isotropy group of $\bar{a}$ is $\Gamma_{\bar{a}}\times S^1$ with

$$\Gamma_{\bar{a}}=\mathbb{Z}_2\times\tilde{\mathbb{Z}}_n.$$

3 The Liapunov-Schmidt reduction

The orthogonal degree was defined only in finite dimension. There is a theoretical difficulty for the extension to abstract infinite dimensional spaces, in particular, for strongly indefinite problems, in the sense that the degrees defined on finite dimensional approximations do not stabilize, as the classical Leray-Schauder degree, when increasing the dimension, although the changes are easy to compute: see, for instance, the case of hamiltonian systems treated in [24], Lemma 3.6, p. 264. In the present situation, one has a very simple way of overcoming this difficulty, without any loss of information and, at the same time, avoiding many technical problems of functional analysis.

Thus, we must perform first a reduction to some finite space .

The bifurcation operator f, in Fourier series, is

$$f(x)=\sum_{l\in\mathbb{Z}}(l^2\nu^2\mathcal{M}x_l-2l\nu\sqrt{\omega}(i\bar{\mathcal{J}})\mathcal{M}x_l+g_l)e^{ilt},$$

where x_l and g_l are the Fourier modes of x and $\nabla V(x)$, which is relatively compact, by Sobolev embedding, with respect to the second order terms. Since all the masses are non-zero, then the matrix $l^2\nu^2\mathcal{M}I-2il\nu\sqrt{\omega}\bar{\mathcal{J}}\mathcal{M}$ is invertible for all large $l\nu$'s, for $\nu>0$. Therefore, if we choose a big enough p, we can solve the modes $|l|>p$ in terms of the remaining $2p+1$ Fourier modes, on any bounded subset of $\mathcal{W}$ with frequencies ν uniformly bounded from below, with an application of the global implicit function theorem, [24], p. 257.

In this way, we get that the zeros of the bifurcation operator f correspond to the zeros of the bifurcation function

$$f(x_1,x_2(x_1,\nu),\nu)=\sum_{|l|\le p}(l^2\nu^2\mathcal{M}x_l-2l\nu\sqrt{\omega}(i\bar{\mathcal{J}})\mathcal{M}x_l+g_l)e^{ilt},$$

where x_1 corresponds to the $2p+1$ Fourier modes and x_2 to the solution for the remaining modes. Then, the linearization of the bifurcation function at some

equilibrium x_0 is

$$f'(x_0) = \sum_{|l|\leq p} \left(l^2\nu^2\mathcal{M} - 2l\nu\sqrt{\omega}(i\bar{\mathcal{J}})\mathcal{M} + D^2V(x_0)\right) x_l e^{ilt}.$$

Since the bifurcation operator is real, the linearization is determined by the blocks $M(l\nu)$ for $l \in \{0, ..., p\}$, where $M(\nu)$ is the matrix

$$M(\nu) = \nu^2\mathcal{M} - 2\nu\sqrt{\omega}(i\bar{\mathcal{J}})\mathcal{M} + D^2V(x_0). \tag{6}$$

Each one of the blocks $M(l\nu)$ corresponds to a Fourier mode:since the orbits are real, one has that the x_l, for negative l's are the conjugates of x_{-l}. This is why we are considering $x_l \in \mathbb{C}$, for $l \in \{0, ..., p\}$. Since we restrict the operator $f(x)$ to the subspace of $H^2_{2\pi}(\mathbb{R}^{3n}\backslash\Psi)$ orthogonal to e, then the block for the first Fourier mode $M(0)$ must be restricted to the orthogonal space to e. We shall denote this restriction as $M(0)^{\perp e}$.

4 Irreducible representations

The following step in order to use the orthogonal degree, [24] Theorem 3.1, p. 247, consists in identifying the irreducible representations, that is, for the action of the isotropy group $\Gamma_{\bar{a}}$ and obtaining the decomposition of the matrix $M(\nu)$.

4.1 The general relative equilibria

We show first, for n arbitrary bodies in the plane, how the matrix $M(\nu)$ decomposes into two blocks by the action of the isotropy group $\Gamma_{x_0} = \mathbb{Z}_2$. The two blocks correspond to the planar and spatial spectra as in the previous decomposition of $D^2V(x_0)$.

Since the group $\mathbb{Z}_2$ acts by reflection on the z-axis, then the equivalent irreducible representations for $\mathbb{C}^{3n}$, leading to the decomposition of $M(\nu)$, are

$$V_0 = \{(x_1, y_1, 0, ..., x_n, y_n, 0) : x_j, y_j \in \mathbb{C}\} \text{ and}$$
$$V_1 = \{(0, 0, z_1, ..., 0, 0, z_n) : z_j \in \mathbb{C}\}.$$

The action of $\kappa \in \mathbb{Z}_2$ on V_0 is given by $\rho(\kappa) = I$ and on V_1 by $\rho(\kappa) = -I$. Define the isomorphisms $T_0 : \mathbb{C}^{2n} \to V_0$ and $T_1 : \mathbb{C}^n \to V_1$ as

$$T_0(x_1, y_1, ..., x_n, y_n) = (x_1, y_1, 0, ..., x_n, y_n, 0),$$
$$T_1(z_1, ..., z_n) = (0, 0, z_1, ..., 0, 0, z_n),$$

and define the linear orthogonal map P

$$P(x_1, y_1, z_1, ..., x_n, y_n, z_n) = T_0(x_1, y_1, ..., x_n, y_n) + T_1(z_1, ..., z_n).$$

The transformation P rearranges the planar and spatial coordinates.

Since V_0 and V_1 are subspaces of equivalent irreducible representations, Schur's lemma implies that the matrix $M(\nu)$ must satisfy

$$P^{-1}M(\nu)P = diag(M_0(\nu), M_1(\nu)),$$

where M_0 and M_1 are respectively $2n \times 2n$ and $n \times n$ matrices. In fact, we will exhibit explicitly the matrices M_0 and M_1, in the following result:

Proposition 10 *Define the matrices $\mathcal{M}_1 = (m_1, ..., m_n)$, $\mathcal{M}_2 = diag(m_1, m_1, ..., m_n, m_n)$, and $\mathcal{J} = diag(J, ..., J)$, then the blocks M_0 and M_1 are*

$$M_0(\nu) = \nu^2\mathcal{M}_2 - 2\nu\sqrt{\omega}(i\mathcal{J})\mathcal{M}_2 + (A_{ij})_{ij=1}^n,$$
$$M_1(\nu) = \nu^2\mathcal{M}_1 + (a_{ij})_{ij=1}^n.$$

Proof. Since P rearranges the planar and spatial coordinates and $D^2V(x_0) = (\mathcal{A}_{ij})_{ij=1}^n$, with $\mathcal{A}_{ij} = diag(A_{ij}, a_{ij})$, then

$$P^{-1}D^2V(x_0)P = diag((A_{ij})_{ij=1}^n, (a_{ij})_{ij=1}^n).$$

Moreover,

$$P^{-1}\mathcal{M}P = diag(\mathcal{M}_2, \mathcal{M}_1) \text{ and } P^{-1}\bar{\mathcal{J}}P = diag(\mathcal{J}, 0).$$

Therefore,

$$P^{-1}M(\nu)P = \nu^2 diag(\mathcal{M}_2, \mathcal{M}_1) - 2\nu\sqrt{\omega} diag((i\mathcal{J})\mathcal{M}_2, 0) + diag((A_{ij})_{ij=1}^n, (a_{ij})_{ij=1}^n).$$

■

The block $M(0)$ must be restricted to the space orthogonal to e. Since $e \in V_1$, with $T_1(1, ..., 1) = e$, then, in the new coordinates, the block $M_1(0)$ must be restricted to the space orthogonal to $(1, ..., 1)$.

For the study of the planar n-body problem, this procedure gives the matrix $M_0(\nu)$ instead of $M(\nu)$.

Planar representation

The action of the isotropy group $\mathbb{Z}_2 \times S^1$ on the space V_0 is given by

$$(\kappa, \varphi)x = e^{i\varphi}x.$$

Since $\kappa \in \mathbb{Z}_2$ fixes the points of V_0, then the space V_0 has its isotropy subgroup generated by κ,

$$\mathbb{Z}_2 = \langle \kappa \rangle .$$

Thus, solutions $x(t)$ with isotropy group $\mathbb{Z}_2$ must satisfy

$$x(t) = \kappa x(t) = Rx(t).$$

Consequently, orbits in V_0 are planar, $z_j(t) = 0$.

4.1.1 Spatial representation

In V_1 the action of the group $\mathbb{Z}_2 \times S^1$ is

$$(\kappa, \varphi)x = -e^{i\varphi}x.$$

Since the action of (κ, π) fixes the points of V_1, then the space V_1 has its isotropy subgroup generated by (κ, π),

$$\tilde{\mathbb{Z}}_2 = \langle (\kappa, \pi) \rangle .$$

Solutions $x(t)$ with isotropy group $\tilde{\mathbb{Z}}_2$ must satisfy

$$x(t) = (\kappa, \pi)x(t) = Rx(t + \pi).$$

Consequently, orbits in V_1 satisfy the symmetry

$$x_j(t) = x_j(t + \pi),\ y_j(t) = y_j(t + \pi) \text{ and } z_j(t) = -z_j(t + \pi). \tag{7}$$

From the symmetry (7), we have that $z_j(t_j) = 0$ for some t_j. This means that these solutions oscillate around the (x, y)-plane and that the projection of the curve in the (x, y)-plane is π-periodic. Furthermore, these solutions go around the projected π-periodic curve once with the spatial coordinate $z_j(t)$ and once with $-z_j(t)$. These solutions look like eights around the equilibrium points, and we shall call them spatial eights.

4.2 The polygonal equilibrium

In the case of the polygonal equilibrium, the isotropy group has also the action of $\tilde{\mathbb{Z}}_n$. Notice that the action of $(\zeta, \zeta) \in \tilde{\mathbb{Z}}_n$ in V_k, $k = 0, 1$, is given by

$$(\zeta, \zeta)T_0(x_0, y_0, ..., x_n, y_n) = T_0(e^{\mathcal{J}\zeta}(x_0, y_0, x_{\zeta(1)}, ..., y_{\zeta(n)})),$$
$$(\zeta, \zeta)T_1(z_0, ..., z_n) = T_1(z_0, z_{\zeta(1)}, ..., z_{\zeta(n)}).$$

Thus, we expect a decomposition of the space V_k into smaller irreducible representations due to the action of $\tilde{\mathbb{Z}}_n$.

4.2.1 Planar representations

We give first the decomposition of the space V_0 into smaller irreducible representations. Since the representations for $n = 2$ are different from those for $n \geq 3$, we shall restrict the study to the case $n \geq 3$ and we shall give some comments on the case $n = 2$ at the end of the paper.

Definition 11 *For $k \in \{2, ..., n - 2, n\}$, define the isomorphisms $T_k : \mathbb{C}^2 \to W_k$ as*

$$T_k(w) = (0, n^{-1/2}e^{(ikI+J)\zeta}w, ..., n^{-1/2}e^{n(ikI+J)\zeta}w) \text{ with}$$
$$W_k = \{(0, e^{(ikI+J)\zeta}w, ..., e^{n(ikI+J)\zeta}w) : w \in \mathbb{C}^2\}.$$

For $k \in \{1, n-1\}$, define the isomorphism $T_k : \mathbb{C}^3 \to W_k$ as

$$T_k(\alpha, w) = (v_k \alpha, n^{-1/2} e^{(ikI+J)\zeta} w, ..., n^{-1/2} e^{n(ikI+J)\zeta} w) \text{ with}$$
$$W_k = \{(v_k \alpha, e^{(ikI+J)\zeta} w, ..., e^{n(ikI+J)\zeta} w) : \alpha \in \mathbb{C}, w \in \mathbb{C}^2\},$$

where v_1 and v_{n-1} are the vectors

$$v_1 = 2^{-1/2}(1, i) \text{ and } v_{n-1} = 2^{-1/2}(1, -i).$$

We have proved, in the paper [19], p.3207, that the subspaces W_k form the irreducible representations of $\tilde{\mathbb{Z}}_n$ in $\mathbb{C}^{2(n+1)} \simeq V_0$. Also, we proved, in [19], Proposition 5, that the action of $(\zeta, \zeta, \varphi) \in \tilde{\mathbb{Z}}_n \times S^1$ on W_{kl}, for the l-th Fourier mode, is given by

$$\rho(\zeta, \zeta, \varphi) = e^{ik\zeta} e^{il\varphi}.$$

Consequently, the isotropy subgroup of $\Gamma_{\bar{a}} \times \mathbb{S}^1$ for the block W_k, for the fundamental mode, is generated by $\kappa \in \mathbb{Z}_2$ and $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n \times S^1$, that is,

$$\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2 = \langle (\zeta, \zeta, -k\zeta) \rangle \times \langle \kappa \rangle.$$

Since the subspaces W_k are orthogonal to each other, then the linear map

$$Pw = \sum_{k=1}^{n} T_k(w_k)$$

is orthogonal, where $w = (w_1, ..., w_n)$, with $w_k \in \mathbb{C}^3$ for $k = 1, n-1$ and $w_k \in \mathbb{C}^2$ for the remaining k's.

Since the map P rearranges the irreducible representations of $\tilde{\mathbb{Z}}_n$, from Schur's lemma, we must have that

$$P^{-1}(A_{ij})_{ij=1}^{n} P = diag(B_1, ..., B_n),$$

where B_k are matrices such that $(A_{ij})_{ij=1}^{n} T_k(w) = T_k(B_k w)$. In fact, in the paper [19], Propositions 18 and 19, we gave the blocks B_k for the general case $\alpha \geq 1$. In the next proposition, we state this result.

Proposition 12 *Set $\alpha_+ = (\alpha+1)/2$ and $\alpha_- = (\alpha-1)/2$. Define*

$$\alpha_k = \alpha_-(s_{k+1} + s_{k-1})/2, \; \beta_k = \alpha_+(s_k - s_1) \text{ and } \gamma_k = \alpha_-(s_{k+1} - s_{k-1})/2$$

with

$$s_k = \frac{1}{2^\alpha} \sum_{j=1}^{n-1} \frac{\sin^2(kj\zeta/2)}{\sin^{\alpha+1}(j\zeta/2)}.$$

Then, the blocks satisfy $B_{n-k} = \bar{B}_k$ and they are given by

$$B_k = \alpha_+(I+R)\mu + (s_1 + \alpha_k)I - \beta_k R - \gamma_k iJ,$$

for $k \in \{2, ..., n-2, n\}$, and

$$B_1 = \begin{pmatrix} \mu(s_1 + \mu + n\alpha_-) & -\sqrt{n/2}\mu\alpha & -\sqrt{n/2}\mu i \\ -\sqrt{n/2}\mu\alpha & s_1 + \alpha_1 + (\alpha+1)\mu & \alpha_1 i \\ \sqrt{n/2}\mu i & -\alpha_1 i & s_1 + \alpha_1 \end{pmatrix}.$$

Since $M_1(\nu)$ is $\tilde{\mathbb{Z}}_n$-equivariant, from Schur's lemma, we have that

$$P^{-1}M_0(\nu)P = diag(m_{01}(\nu), ..., m_{0n}(\nu)),$$

where the matrices $m_{0k}(\nu)$ must satisfy $M_0 T_k(w) = T_k(m_{0k}w)$.

Proposition 13 *The matrices $m_{0k}(\nu)$ satisfy $m_{0k}(\nu) = \bar{m}_{0(n-k)}(-\nu)$ with*

$$m_{0k}(\nu) = \nu^2 I - 2\nu\sqrt{\omega}(iJ) + B_k$$

for $k \in \{2, ..., n-2, n\}$, and

$$m_{01}(\nu) = \nu^2 diag(\mu, I) - 2\nu\sqrt{\omega} diag(\mu, iJ) + B_1.$$

Proof. The matrix $\mathcal{M}_2$ is $diag(\mu, \mu, 1, ..., 1)$. For $k \in \{2, ..., n-2, n\}$, the matrix $\mathcal{M}_2$ satisfies $\mathcal{M}_2 T_k(w) = T_k(w)$. Since $(i\mathcal{J})T_k(w) = T_k(iJw)$, then

$$M_0 T_k(w) = \left(\nu^2\mathcal{M}_2 - 2\nu\sqrt{\omega}\mathcal{M}_2(i\mathcal{J}) + (A_{ij})_{ij=1}^n\right) T_k(w) = T_k(m_{0k}w).$$

For $k = 1$, the matrix $\mathcal{M}_2$ satisfies $\mathcal{M}_2 T_1(w) = T_1(diag(\mu, 1, 1)w)$. Since $(iJ)v_1 = v_1$, then $(i\mathcal{J})T_1(w) = T_1(diag(1, iJ)w)$. Therefore,

$$M_0 T_1(w) = \left(\nu^2\mathcal{M}_2 - 2\nu\sqrt{\omega}\mathcal{M}_2(i\mathcal{J}) + (A_{ij})_{ij=1}^n\right) T_1(w) = T_1(m_{01}w).$$

For $k = n-1$, using $(iJ)v_2 = -v_2$ and a similar argument, we may prove that

$$m_{0(n-1)}(\nu) = \nu^2 diag(\mu, I) - 2\nu\sqrt{\omega} diag(-\mu, iJ) + B_{n-1}.$$

The equalities $m_{0(n-k)}(\nu) = \bar{m}_{0k}(-\nu)$ follow from $B_{n-k} = \bar{B}_k$. ■

Remark 14 *In fact, the equality $m_{0(n-k)}(\nu) = \bar{m}_{0k}(-\nu)$ is a consequence of the $\tilde{\mathbb{Z}}_2$-equivariant property of the operator by the action $\tilde{\kappa}x_j(t) = \mathcal{R}_1 x_{n-j}(-t)$, for $j = 1, ..., n-1$, and $\tilde{\kappa}x_j(t) = \mathcal{R}_1 x_j(-t)$, for $j = 0$ and $j = n$.*

In fact the bifurcation operator $f(x)$ is equivariant under the action of the group

$$(\mathbb{Z}_n \times T^2) \times \tilde{\kappa}(\mathbb{Z}_n \times T^2),$$

where the action of $\tilde{\kappa}$ is given, for the n equal mass bodies, modulus n, by

$$\tilde{\kappa}x_j(t) = R_1 x_{n-j}(-t) \text{ with } R_1 = diag(1, -1, 1),$$

while, for the central body, $j = 0$, the action is

$$\tilde{\kappa}x_0(t) = R_1 x_0(-t).$$

Let z_k be the coordinate for the block m_{0k}. The action of $\tilde{\kappa}$ on (z_k, z_{n-k}) is given by

$$\tilde{\kappa}(z_k, z_{n-k}) = (R\bar{z}_{n-k}, R\bar{z}_k),$$

where $R = diag(1, -1)$ for $k \notin \{1, n-1\}$ and $R = diag(1, 1, -1)$ for $k \in \{1, n-1\}$, and $\tilde{\kappa}(z_n) = R\bar{z}_n$.

Now, since the polygonal equilibrium $\bar{a}$ is fixed by $\tilde{\kappa}$, then the linearization of the bifurcation map $f(x)$, represented by the blocks m_{0k}, must be $\langle\tilde{\kappa}\rangle$ -equivariant. This means that

$$\tilde{\kappa} diag(m_{0k}, m_{0(n-k)})(z_k, z_{n-k}) = diag(m_{0k}, m_{0(n-k)})\tilde{\kappa}(z_k, z_{n-k}).$$

From the first coordinate one gets the equality $R\bar{m}_{0(n-k)} = m_{0k}R$, or equivalently

$$m_{0k} = R\bar{m}_{0(n-k)}R,$$

which is equivalent to the equality $m_{0k}(\nu) = \bar{m}_{0(n-k)}(-\nu)$.

Now, if we restrict the bifurcation operator $f(x)$ to the fixed-point space of $\tilde{\kappa}$, then we get that the coordinates z_k of the blocks m_{0k} are related by

$$z_{n-k} = R\bar{z}_k \text{ and } z_n = R\bar{z}_n.$$

Using the isomorphism $Tz_k = (z_k, R\bar{z}_k)$ for $k \notin \{n/2, n\}$ and $T(x_k, y_k) = (x_k, iy_k) \in \mathbb{C}^2$ for $k \in \{n/2, n\}$, one may see that $M_0(\nu)$, in the fixed-point space of $\tilde{\kappa}$ is equivalent to the matrix

$$(m_{01}, m_{02}, m_{03}, ..., m_{0(n/2)}, m_{0n})$$

defined in the space

$$\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2 \times ... \times \mathbb{R}^2 \times \mathbb{R}^2.$$

Applying degree in the fixed-point space of $\tilde{\kappa}$, one proves bifurcation of periodic solutions for the blocks given by m_{0n} and $m_{0n/2}$. Moreover, since the action of κ and

$$(\zeta, \zeta, 0) \in \mathbb{Z}_n \times S^1 \times S^1$$

on m_{0n} is trivial, as well as the action of κ and (ζ, ζ, π) on $m_{0n/2}$, then one can deduce also that these equilibria will have the symmetries that we described above.

The remaining blocks m_{0k} for $k \notin \{n/2, n\}$ are defined in complex subspaces, then they have always non negative determinants, as real matrices. Thus, there is no change of a standard degree in the isotropy subspace. Analytical studies with normal forms of high order and additional hypotheses of non-resonance are proposed in [8] for these cases. One could also use the gradient structure and apply the results for bifurcation given in [21], p.100, based on Conley index. However, these results do not provide the proof of the existence of a continuum, something which follows from the application of the orthogonal degree.

4.2.2 Spatial representation

We analyze now the decomposition of V_1.

Definition 15 *For $k \in \{1, ..., n-1\}$, we define $T_k : \mathbb{C} \to W_k$ as*

$$T_k(w) = (0, n^{-1/2}e^{ik\zeta}w, ..., n^{-1/2}e^{nik\zeta}w) \text{ with}$$
$$W_k = \{(0, e^{ik\zeta}w, ..., e^{nik\zeta}w) : w \in \mathbb{C}\}.$$

And, for $k = n$, we define $T_n : \mathbb{C}^2 \to W_n$ as

$$T_n(\alpha, w) = (\alpha, n^{-1/2}w, ..., n^{-1/2}w) \text{ with}$$
$$W_n = \{(\alpha, w, ..., w) : \alpha, w \in \mathbb{C}\}.$$

The action of (ζ, ζ) in the subspace W_k is given by

$$(\zeta, \zeta)T_k(w) = T_k(e^{ik\zeta}w).$$

Consequently, the decomposition in irreducible representations of the space $\mathbb{C}^{n+1} \simeq V_1$, for the group $\tilde{\mathbb{Z}}_n$, are the subspaces W_k. The actions of the elements $\kappa \in \mathbb{Z}_2$, $(\zeta, \zeta) \in \tilde{\mathbb{Z}}_n$ and $\varphi \in S^1$ in W_k are

$$\rho(\kappa) = -1, \rho(\zeta, \zeta) = e^{ik\zeta} \text{ and } \rho(\varphi) = e^{il\varphi}.$$

Since the elements $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n \times S^1$ and $(\kappa, \pi) \in \mathbb{Z}_2 \times S^1$ act trivially on W_k, then the isotropy group of W_k is generated by $(\zeta, \zeta, -k\zeta)$ and (κ, π),

$$\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2 = \langle(\zeta, \zeta, -k\zeta)\rangle \times \langle(\kappa, \pi)\rangle. \tag{8}$$

Since the spaces W_k are orthogonal to each other, then the linear map

$$Pw = \sum_{k=1}^{n} T_k(w_k)$$

is orthogonal, where $w = (w_1, ..., w_n)$, with $w_n \in \mathbb{C}^2$ and $w_k \in \mathbb{C}^1$ for the remaining k's.

Since $M_1(\nu)$ is $\tilde{\mathbb{Z}}_n$-equivariant, by Schur's lemma, the matrix $M_1(\nu)$ must satisfy

$$P^{-1}M_1(\nu)P = diag(m_{11}(\nu), ..., m_{1n}(\nu)),$$

where the blocks $m_{1k}(\nu)$ are such that $M_1T_k(w) = T_k(m_{1k}w)$ for $k \in \{1, ..., n\}$.

Proposition 16 *We have $m_{1k}(\nu) = \nu^2 - (\mu + s_k)$ for $k \in \{1, ..., n-1\}$, and*

$$m_{1n}(\nu) = \begin{pmatrix} \mu(\nu^2 - n) & \sqrt{n}\mu \\ \sqrt{n}\mu & \nu^2 - \mu \end{pmatrix}.$$

Proof. We denote the coordinate $w_i \in \mathbb{R}^2$ of the vector $w = (w_0, ..., w_n)$ by $[w]_i$. For $k \in \{1, ..., n-1\}$, if $l \neq 0$, from the definition

$$[(a_{ij})T_k(w)]_l = n^{-1/2} \sum_{j=1}^{n} a_{lj}e^{ijk\zeta}w.$$

Since $a_{lj} = m_l m_j / d_{lj}$, with $d_{lj}^2 = 4\sin^2((l-j)\zeta/2)$ for $l, j \in \{1, ..., n\}$, then $a_{lj} = a_{n(j-l)}$, with $(j-l) \in \{1, ..., n\}$, modulus n. From the equality $a_{lj} e^{ijk\zeta} = e^{ilk\zeta}\left(a_{n(j-l)} e^{i(j-l)k\zeta}\right)$, we have that

$$[(a_{ij}) T_k(w)]_l = n^{-1/2} e^{ilk\zeta} \left(\sum_{j=1}^{n} a_{nj} e^{ijk\zeta} \right) w = [T_k(b_k w)]_l \,,$$

where b_k is the sum between parentheses.

In order to calculate b_k, notice that $a_{nn} = -\sum_{j=0}^{n-1} a_{nj}$ and $a_{n0} = \mu$, then

$$b_k = -\mu + \sum_{j=1}^{n-1} \left(e^{ijk\zeta} - 1\right) a_{nj}.$$

Since $a_{nj} = 1/d_{nj}^{\alpha+1}$, then

$$\sum_{j=1}^{n-1} \left(e^{ijk\zeta} - 1\right) a_{nj} = -\sum_{j=1}^{n-1} \frac{2\sin^2(kj\zeta/2)}{2^{\alpha+1}\sin^{\alpha+1}(j\zeta/2)} = -s_k.$$

Thus, we have $(a_{ij})T_k(w) = T_k(b_k w)$, with $b_k = -(\mu + s_k)$.

If $l \neq 0$, then,

$$[(a_{ij}) T_n(\alpha, w)]_l = a_{l0}\alpha + n^{-1/2} \sum_{j=1}^{n} a_{lj} w,$$

with $\alpha, w \in \mathbb{C}$. Using the equalities $a_{l0} = \mu$ and $\sum_{j=1}^{n} a_{lj} = -a_{l0}$, we have

$$[(a_{ij}) T_n(\alpha, w)]_l = n^{-1/2}(\sqrt{n}\mu\alpha - \mu w). \tag{9}$$

If $l = 0$, then

$$[(a_{ij}) T_n(\alpha, w)]_0 = a_{00}\alpha + n^{-1/2} \sum_{j=1}^{n} a_{0j} w.$$

Using the equality $a_{00} = -n\mu$, we have

$$[(a_{ij}) T_n(\alpha, w)]_0 = -n\mu\alpha + \sqrt{n}\mu w. \tag{10}$$

From the equalities (9) and (10), we conclude that

$$(a_{ij}) T_n(\alpha, w) = T_n\left(\begin{pmatrix} -n\mu & \sqrt{n}\mu \\ \sqrt{n}\mu & -\mu \end{pmatrix} \begin{pmatrix} \alpha \\ w \end{pmatrix} \right).$$

Here the matrix $\mathcal{M}_1$ is $diag(\mu, 1, ..., 1)$. Then, the statements of the proposition follow from the fact that $\mathcal{M}_1 T_k(w) = T_k(w)$ for $k \in \{1, ..., n-1\}$, and $\mathcal{M}_1 T_n(\alpha, w) = T_n(\mu\alpha, w)$. ■

Since the block $M_1(0)$ is restricted to the space orthogonal to $(1, ..., 1)$, and since $T_n(1, \sqrt{n}) = (1, ..., 1)$, then the block $m_{1n}(0)$ must be restricted to the subspace orthogonal to $(1, \sqrt{n})$.

As we expected, the eigenvalues of $m_{1n}(0)$ are 0 and $-(n+1)\mu$ with eigenvectors $(1, \sqrt{n})$ and $(\sqrt{n}, -1)$ respectively. Therefore, the matrix $m_{1n}(0)$, on the subspace orthogonal to $(1, \sqrt{n})$, is equivalent to

$$m_{1n}(0)^{\perp} = -(n+1)\mu.$$

The map $m_{1n}(0)^{\perp}$ is invertible for $\mu \neq 0$. For $\mu = 0$ the central body has zero mass, and, in this case, the coordinate corresponding to $m_{1n}(0)^{\perp}$ may be taken away.

Remark 17 *In the case of the spatial blocks, let z_k be the the coordinate of the block m_{1k}.The action on this coordinate is given by*

$$\tilde{\kappa}(z_k, z_{n-k}) = (\bar{z}_{n-k}, \bar{z}_k),$$

for $k = 1, ..., n-1$ and $\tilde{\kappa} z_n = \bar{z}_n$.

Since the blocks m_{1k} must be $\langle \tilde{\kappa} \rangle$ -equivariant, then one gets the equality

$$m_{1k} = \bar{m}_{1(n-k)}.$$

Thus, when restricting the bifurcation operator $f(x)$ to the fixed-point space of $\tilde{\kappa}$, the coordinates z_k of the blocks m_{1k} are related by $z_{n-k} = \bar{z}_k$, and $z_n = \bar{z}_n$.

Using the isomorphism $Tz_k = (z_k, \bar{z}_k)$ for $k \notin \{n/2, n\}$, the inclusion of $\mathbb{R}$ in $\mathbb{C}$ for $k = n/2$, and the inclusion of $\mathbb{R}^2$ in $\mathbb{C}^2$ for $k = n$, one may prove that $M_1(\nu)$ in the fixed-point space of $\tilde{\kappa}$, is equivalent to the matrix

$$(m_{11}, m_{12}, ..., m_{1(n/2)}, m_{1n}),$$

defined in the space

$$\mathbb{C} \times \mathbb{C} \times ... \times \mathbb{R} \times \mathbb{R}^2.$$

Applying a degree argument in the fixed-point space of $\tilde{\kappa}$, one may prove bifurcation of periodic solutions for the blocks given by m_{1n} and $m_{1n/2}$. The remaining blocks m_{0k} for $k \notin \{n/2, n\}$ are defined on complex subspaces, then they have always non-negative determinants, as real matrices. Using the comments of the previous remark, one may give a different proof of the results on vertical bifurcation discussed in [8].

5 Symmetries

Before we show the existence of bifurcation points, we wish to describe the symmetries of the solutions. That is, solutions with isotropy group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2$ for the blocks $m_{0k}(\nu)$ and $\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2$ for the blocks $m_{1k}(\nu)$.

The coordinates (x_j, y_j, z_j), of the body $j \in \{0, ..., n\}$, will denote the components of the solution $x(t)$.

5.1 Planar solutions

In this part we describe the symmetries of the isotropy group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2$. As we have seen, due to the group $\mathbb{Z}_2$, these solutions must be planar. We shall identify the real and complex planes by $u_j = x_j + iy_j$.

Now, since $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n(k)$, the solutions with isotropy group $\tilde{\mathbb{Z}}_n(k)$ must satisfy

$$u_j(t) = e^{-i\zeta} u_{\zeta(j)}(t - k\zeta).$$

Remark 18 *Notice that, if $u_j(t)$ is a solution, for ν, with symmetry $\tilde{\mathbb{Z}}_n(k)$, then $u_j(-t)$ is a solution, for $-\nu$, with symmetry $\tilde{\mathbb{Z}}_n(n-k)$. In fact, the bifurcation phenomena are related by the fact that $m_{0(n-k)}(\nu) = \bar{m}_{0k}(-\nu)$.*

In order to describe the symmetries of the group $\tilde{\mathbb{Z}}_n(k)$ we need the following definition.

Definition 19 *For each fixed k, let h be the maximum common divisor of n and k. We define*

$$\bar{n} = n/h \text{ and } \bar{k} = k/h.$$

For the central body u_0 one has the following symmetries.

Proposition 20 *If $h > 1$, the central body remains at the center $u_0(t) = 0$. If $h = 1$, the central body satisfies*

$$u_0(t + \zeta) = e^{-ik'\zeta} u_0(t),$$

were k' is such that $k'k = 1$, modulus n.

Proof. Since $\zeta(0) = 0$, the central body has the symmetry

$$u_0(t) = e^{-i(l\zeta)} u_0(t - k(l\zeta)).$$

For $h > 1$ take $l = \bar{n}$. Since $k(\bar{n}\zeta) = 2\pi\bar{k}$ and $\bar{n}\zeta = 2\pi/h$, then the central body satisfies

$$u_0(t) = e^{-i(\bar{n}\zeta)} u_0(t - k(\bar{n}\zeta)) = e^{-i(2\pi/h)} u_0(t) = 0.$$

For $h = 1$ take $l = k'$, then the central body satisfies

$$u_0(t) = e^{-i(k'\zeta)} u_0(t - k(k'\zeta)) = e^{-i(k'\zeta)} u_0(t - \zeta).$$

■

In order to describe the symmetries of the n bodies with equal masses, we use the notation $u_j = u_{j+kn}$, for $j \in \{1, ..., n\}$. Then, $\zeta(j) = j + 1$, and the n bodies satisfy

$$u_{j+1}(t) = e^{\,ij\zeta} u_1(t + jk\zeta).$$

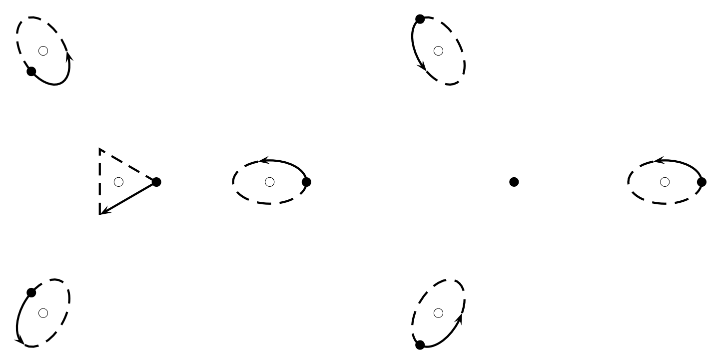
Thus, each one of the n bodies follows the same planar curve, but with a different phase and with some rotation in the (x, y)-plane.

Now, let us show some examples of these symmetries.

For $\tilde{\mathbb{Z}}_n(n)$, the symmetries are

$$u_0(t) = 0 \text{ and } u_{j+1}(t) = e^{\,ij\zeta} u_1(t).$$

Thus, the central body remains at the center, and the other n bodies form a n-polygon at any time: see the figure for $n = 3$.



(a) Symmetries of $\tilde{\mathbb{Z}}_n(1)$. (b) Symmetries of $\tilde{\mathbb{Z}}_n(3)$.

Figure 1: For $n = 3$.

For $\tilde{\mathbb{Z}}_n(1)$, the symmetries are

$$u_0(t+\zeta) = e^{-i\zeta} u_0(t) \text{ and } u_{j+1}(t) = e^{\,ij\zeta} u_1(t + j\zeta).$$

Therefore, the central body is determined by the time interval $[0, \zeta)$, and the rest of its orbit is given by rotations. The other n bodies follow the orbit of one of them, but with a synchronization between the change of phase and the (x, y)-rotation: see the figure for $n = 3$.

For a group $\tilde{\mathbb{Z}}_n(k)$, with $h = 1$, the symmetries are

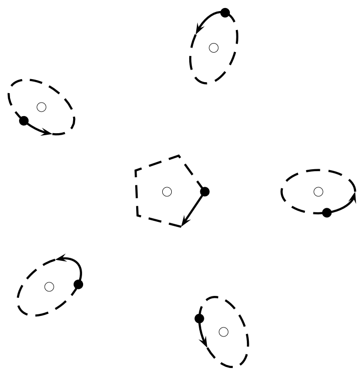
$$u_0(t+\zeta) = e^{-ik'\zeta} u_0(t) \text{ and } u_{j+1}(t) = e^{\,ij\zeta} u_1(t + j(k\zeta)).$$

These symmetries are similar to the ones for $\tilde{\mathbb{Z}}_n(1)$, but now there is a permutation between the phase and the planar rotation. For instance, one may compare the cases $\tilde{\mathbb{Z}}_5(1)$ and $\tilde{\mathbb{Z}}_5(2)$, for $n = 5$.

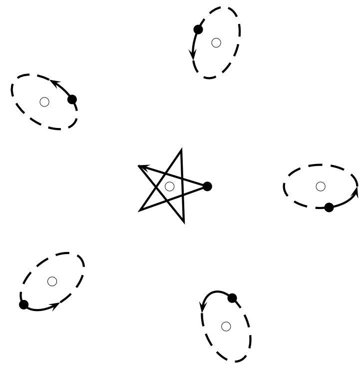
The effect of the group $\tilde{\mathbb{Z}}_5(1)$ was already described. In order to show the symmetries for the group $\tilde{\mathbb{Z}}_5(2)$, notice that $k' = 3$, then the central body visits the points: $u_0(\zeta) = e^{-3i\zeta} u_0$, $u_0(2\zeta) = e^{-i\zeta} u_0$, $u_0(3\zeta) = e^{-4i\zeta} u_0$ and $x_0(4\zeta) = e^{-2i\zeta} u_0$. The other n bodies have orbits: $u_1(t)$, $e^{\,i\zeta} u_1(t + 2\zeta)$, $e^{\,i2\zeta} u_1(t + 4\zeta)$, $e^{\,i3\zeta} u_1(t + \zeta)$ and $e^{\,i4\zeta} u_1(t + 3\zeta)$.

For the group $\tilde{\mathbb{Z}}_n(k)$ with $h > 1$, the symmetries are

$$u_0(t) = 0 \text{ and } u_{j+1}(t) = e^{\,ij\zeta} u_1(t + j\bar{k}(2\pi/\bar{n})).$$



(a) Symmetries of $\tilde{\mathbb{Z}}_n(1)$.



(b) Symmetries of $\tilde{\mathbb{Z}}_n(2)$.

Figure 2: For $n = 5$.

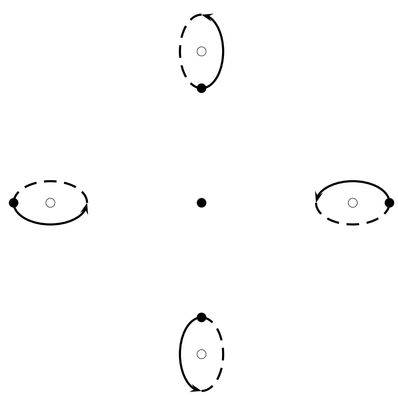


Figure 3: Symmetries of $\tilde{\mathbb{Z}}_n(2)$ for $n = 4$.

In this case the central body remains at the center. Moreover, the other n bodies follow the same curve with a different phase and the planar rotation determined by multiplying by $\bar{k}$ in $\mathbb{Z}_{\bar{n}}$. See the example $n = 4$ and $k = 2$.

Remark 21 *In fixed coordinates, the solutions are $q_j(t) = e^{i\sqrt{\omega}t}u_j(\nu t)$. Reparametrizing the time, we have that $q_j(t) = e^{i\sqrt{\omega}t/\nu}u_j(t)$, where u_j is a 2π-periodic function. Set $\Omega = 1 - k\sqrt{\omega}/\nu$, then, for $j \in \{1, ..., n\}$, we have that*

$$\begin{aligned} q_{j+1}(t) &= e^{it\sqrt{\omega}/\nu}u_{j+1}(t) \\ &= e^{\ ij\zeta}e^{it\sqrt{\omega}/\nu}u_1(t + jk\zeta) = e^{\ ij\zeta\Omega}q_1(t + jk\zeta). \end{aligned}$$

In particular, when the central body has mass zero, we are considering the n-body problem with equal masses. If $\Omega \in n\mathbb{Z}$, the solutions with isotropy group $\tilde{\mathbb{Z}}_n(k)$ satisfy

$$q_{j+1}(t) = q_1(t + jk\zeta).$$

These are solutions where all the bodies follow the same path, and they are known as choreographies, [6].

5.2 Spatial solutions

Now we wish to describe the symmetries of a solution with isotropy group $\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2$. As we saw for the group $\tilde{\mathbb{Z}}_2$, these solutions satisfy

$$u_j(t) = u_j(t+\pi) \text{ and } z_j(t) = -z_j(t+\pi),$$

where $u_j = x_j + iy_j$ is the projection in the (x, y)-plane.

Since the group $\tilde{\mathbb{Z}}_n(k)$ is generated by $(\zeta, \zeta, -k\zeta)$, these solutions satisfy the symmetries $u_j(t) = e^{-i\zeta} u_{\zeta(j)}(t - k\zeta)$ and

$$z_j(t) = z_{\zeta(j)}(t - k\zeta). \tag{11}$$

Therefore, the projection in the (x, y)-plane follows a π-periodic curve with the symmetries of the previous section. The spatial solutions go around this projected curve twice, once with $z_j(t)$ and once with $z_j(-t)$. Hence, the solutions look like spatial eight where the spatial positions satisfy the symmetries (11).

Hence, we need to identify the spatial symmetries (11). For the central body we have that

$$z_0(t) = z_0(t + k\zeta).$$

Using the notation $z_j = z_{j+kn}$ for the other n bodies, their spatial positions are related by

$$z_{j+1}(t) = z_1(t + jk\zeta).$$

Therefore, the spatial curves of the n bodies are determined by just one of them. To see one example, we suppose that $n = 2m$ and choose $k = m$. In this case the central body remains at the center. Moreover, the n bodies with equal masses satisfy

$$u_{j+1}(t) = e^{\,ij\zeta} u_1(t + j\pi) = e^{\,ij\zeta} u_1(t)$$

and

$$z_{j+1}(t) = z_1(t + j\pi) = (-1)^j z_1(t).$$

Thus, there are two m-polygons which oscillate vertically, one with $z_1(t)$ and the other with $-z_1(t)$. Furthermore, the projection of the two m-polygons in the plane is always a $2m$-polygon. These solutions are known as Hip-Hop orbits.

Remark 22 *An element in the intersection of two isotropy subspaces, for k_1 and k_2, will be such that $u_0(t) = 0$, $u_j(t)$, $z_0(t)$, $z_j(t)$ will be $(k_2 - k_1)\zeta$-periodic, that is, if h is the greatest common factor of n and $k_2 - k_1$ and $\tilde{n}h = n$, then the period will be $2\pi/\tilde{n}$.*

6 The bifurcation theorem

The orthogonal degree is defined for orthogonal maps which are non-zero on the boundary of some open bounded invariant set. The degree is made of integers, one for each orbit type, and it has all the properties of the usual Brouwer degree. Hence, if one of the integers is non-zero, then the map has a zero corresponding to the orbit type of that integer. In addition, the degree is invariant under orthogonal deformations that are non-zero on the boundary. The degree has other properties such as sum, products and suspensions, for instance, the degree of two pieces of the set is the sum of the degrees. The interested reader may consult [24], Chapters 2 and 4,[1] and [31] for more details on equivariant degree and degree for gradient maps.

Now, if one has an isolated orbit, then its linearization at one point of the orbit x_0 has a block diagonal structure, due to Schur's lemma, [24], Lemma 7.2, p.30, where the isotropy subgroup of x_0 acts as $\mathbb{Z}_n$ or as S^1. Therefore, the orthogonal index of the orbit is given by the signs of the determinants of the submatrices where the action is as $\mathbb{Z}_n$, for $n = 1$ and $n = 2$, and the Morse indices of the submatrices where the action is as S^1, [24], Theorem 3.1, p. 247. In particular, for problems with a parameter, if the orthogonal index changes at some value of the parameter, one will have bifurcation of solutions with the corresponding orbit type. Here, the parameter is the frequency ν, [24], Proposition 3.1, p. 255.

For a k-dimensional orbit with a tangent space generated by k of the infinitesimal generators of the action of the group, one uses a Poincaré section for the map augmented with k Lagrange-like multipliers for the generators. (See the construction in [24], Section 4.3, p. 245.) For instance, for the action of $SO(2)$, the study of the zeros of the equivariant map $F(x)$, orthogonal to the generator Ax, is equivalent to the study of the zeros of $F(x) + \lambda Ax$, if x is not fixed by the the group, i.e., if Ax is not 0, for which λ is 0. In this way, one has added an artificial parameter. This trick has been used very often and, in the context of a topological degree argument, was called "orthogonal degree" by S. Rybicki in [30]. See also [11] and [22], p. 481, for the case of gradients. The general case of the action of an abelian group was treated in [23]. The complete study of the orthogonal degree theory is given in [24], Chapters 2 and 4.

Any Fourier mode will give rise to an orbit type (modes which are multiples of it), hence one has an element of the orthogonal degree for each mode. Furthermore, if $x(t)$ is a periodic solution, with frequency ν, then $y(t) = x(nt)$ is a $2\pi/n$-periodic solution, with frequency ν/n. Hence, any branch arising from the fundamental mode will be reproduced in the harmonic branch. If one wishes to study period-doubling, then one has to consider the branch corresponding to π-periodic solutions, [24], Corollary 2.1, p.219.

The complete study of the orthogonal degree theory is given in [24], Chapters 2 and 4.

6.1 General relative equilibrium

To use successfully the results of the orthogonal degree from [24], Proposition 3.2, p. 258, we need to verify the hypothesis that the orbit Γx is hyperbolic near a bifurcation point, [24], Definition 2.2, p. 222. In particular, we need to prove that the kernel of $M(0)^{\perp e}$ is generated by $A_1 x_0 = -\bar{\mathcal{J}} x_0$. Since $A_1 x_0 \in V_0$ and $e \in V_1$, an equivalent condition consists in showing that the kernel of $M_0(0)$ is generated by $T_0^{-1}(A_1 x_0)$, and that the kernel of $M_1(0)$ is generated by $T^{-1}(e) = (1, ..., 1)$.

Definition 23 *Following [24], p. 258, we define σ as the sign of the determinant of $M_0(0)$ in the space orthogonal to $T_0^{-1}(\bar{\mathcal{J}} x_0)$.*

Now, if $(z_1, ..., z_n)$ is in the kernel of $M_1(0) = (a_{ij})$, then $z_j = z_i$, because

$$0 = \sum_{j=1}^{n} a_{ij} z_j = \sum_{j=1\ (j \neq i)}^{n} (z_j - z_i) a_{ij}.$$

Consequently, the kernel of $M_1(0)$ is generated by $(1, ..., 1)$, and the hypothesis of [24], Proposition 3.2, p. 258, near a bifurcation point, is assured by the condition $\sigma \neq 0$.

Definition 24 *Let $n_j(\nu)$ be the Morse number of $M_j(\nu)$, for $j = 0, 1$, and define*

$$\eta_j(\nu) = \sigma(n_j(\nu - \rho) - n_j(\nu + \rho)).$$

The number $\eta_j(\nu_0)$ represents the change of the Morse index at the point ν_0. Applying the bifurcation theorems of ([24]), Remark 3.5, p. 259, we get the following result:

Theorem 25 *If $\eta_j(\nu_j)$ is different from zero, the equilibrium x_0 has a global bifurcation of periodic solutions from $2\pi/\nu_j$, with isotropy group $\mathbb{Z}_2$, for $j = 0$, and $\tilde{\mathbb{Z}}_2$, for $j = 1$.*

The bifurcation branch is *non-admissible* when a) the norm or the period goes to infinity or b) the branch ends in a collision path. In any other case, we say that the bifurcation is *admissible.* By global bifurcation we mean that: if the branch is admissible, then the branch must return to other bifurcation points, and the sum of the local degrees at the bifurcation points $\eta_k(\nu)$ must be zero. See [21] for this kind of arguments.

6.2 The polygonal equilibria

We have decomposed the matrix $M(\nu)$ into the two blocks $M_0(\nu)$ and $M_1(\nu)$. Thus, there is a bifurcation of periodic solutions when the Morse index of $M_k(\nu)$ changes.

For the polygonal equilibrium, we have decomposed the blocks $M_0(\nu)$ and $M_1(\nu)$ into $m_{01}(\nu), ..., m_{0n}(\nu)$ and $m_{11}(\nu), ..., m_{1n}(\nu)$ respectively. Actually,

the orthogonal degree for the polygonal equilibrium has one element for each matrix $m_{jk}(\nu)$. Thus, we may prove a more complete result because the group of symmetries is bigger.

Definition 26 *For $j = 0, 1$ and $k = 1, ..., n$, set $n_{jk}(\nu)$ to be the Morse number of $m_{jk}(\nu)$, and define*

$$\eta_{jk}(\nu) = \sigma(n_{jk}(\nu - \rho) - n_{jk}(\nu + \rho)).$$

The fixed point space of the isotropy group $\Gamma_{\bar{a}} \times S^1$ corresponds to the block $m_{0n}(0) = B_n$. Since the generator of the kernel is $A_1\bar{a} = T_n(-n^{1/2}e_2)$, then, in this case, e_2 is in the kernel of $m_{0n}(0)$.

Let σ be the sign of $m_{0n}(0)$ in the space orthogonal to e_2. Since $\beta_n = -\alpha_+ s_1$ and $\alpha_n = \alpha_- s_1$, then $B_n = diag((\alpha + 1)(\mu + s_1), 0)$. Thus, for the polygonal equilibrium, σ is the sign of $(\alpha + 1)(\mu + s_1)$, [19], p. 3220,

$$\sigma = \sigma_n(\mu) = 1.$$

In this case, we still need the hyperbolic condition near a bifurcation point. That is, we need to be sure that the matrices $m_{0k}(0) = B_k$ are invertible for $k = 1, ..., n - 1$. In the paper [19], Proposition 23, we did prove that the block B_k is invertible except for one point $\mu_k \in (s_1, \infty)$. In fact, there is a bifurcation branch of relative equilibria from each μ_k for $k = 1, ..., n - 1$.

From the results of [24], Remark 3.5, p. 259, we may state the following theorem for $\mu \neq \mu_1, ..., \mu_{n-1}$.

Theorem 27 *If $\eta_{jk}(\nu_k)$ is different from zero, then the polygonal equilibrium has a global bifurcation of periodic solutions from $2\pi/\nu_k$, with isotropy group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2$, for $j = 0$, and $\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2$, for $j = 1$.*

7 Spectral analysis

In the previous section, we have proved that there is a bifurcation of periodic solutions when the blocks of $M_k(\nu)$ change their Morse index. The spatial and planar blocks were given in proposition (10). We will analyze the spatial spectrum for a general relative equilibrium, corresponding to $M_1(\nu)$.

For the polygonal equilibrium, the blocks are given in propositions (12) and (16), and the description of the isotropy groups was done in section four. In this section we analyze completely the spectrum of all these blocks $m_{jk}(\nu)$.

7.1 Spatial spectrum for a general equilibrium

The spatial block of a general relative equilibrium is

$$M_1(\nu) = \nu^2 \mathcal{M}_1 + (a_{ij})_{ij=1}^n,$$

were the matrix (a_{ij}) was given in proposition (8)

The following result is well known in linear algebra.

Lemma 28 *Let $B(z_0, r)$ be the ball of center z_0 and radius r. Then, the spectrum $\sigma(A)$ of a matrix $A = (a_{ij})_{ij=1}^n$ is in the union $\bigcup_{i=1}^n B(a_{ii}, r_i)$ with $r_i = \sum_{j=1(j\neq i)}^n |a_{ij}|$.*

The matrix (a_{ij}) has real eigenvalues because it is selfadjoint. And from the previous lemma, we have that the eigenvalues of the matrix (a_{ij}) are in the union $\bigcup_{i=1}^n B(a_{ii}, |a_{ii}|)$, because $a_{ii} = -\sum_{j\neq i} a_{ij}$. Since a_{ii} is negative, then $M_1(0) = (a_{ij})$ must have $n-1$ negative eigenvalues.

Since the eigenvalues of $M_1(\nu)$ are continuous, and $M_1(0)$ has $n-1$ negative eigenvalues, then the Morse number of $M_1(\nu)$ satisfies $n_1(\nu) \geq n-1$ for any small ν. Now, since $n_1(\infty) = 0$, then the matrix $M_1(\nu)$ must change its Morse index at $n-1$ values of ν. However, these $n-1$ values could be the same, and we may assure the existence of only one point where the Morse index changes.

Theorem 29 *For the n-body problem, each relative equilibrium with $\sigma \neq 0$ has at least one global bifurcation branch of periodic solutions with symmetries $\tilde{\mathbb{Z}}_2$. Generically, the equilibrium has $n-1$ bifurcations of periodic eight solutions.*

In the particular case where all the bodies have the same mass, $m_j = m$, let P be a matrix such that $m^{-1}(a_{ij}) = P^{-1}\Lambda P$, where $\Lambda = diag(-\nu_1^2, ..., -\nu_n^2)$ with

$$-\nu_1^2 \leq ... \leq -\nu_{n-1}^2 < \nu_n = 0.$$

Since

$$M_1(\nu) = mP^{-1}(\nu^2 I + \Lambda)P,$$

then the matrix $M_1(\nu)$ has eigenvalues $\lambda_k = m(\nu^2 - \nu_k^2)$.

If all the ν_k's are different, then $n_1(\nu_k - \rho) = k$ and $n_1(\nu_k + \rho) = k-1$. Thus, for $k = 1, ..., n-1$, the change of Morse index of $M_1(\nu)$ at the points ν_k is

$$\eta_1(\nu_k) = \sigma.$$

If the eigenvalue ν_k has multiplicity j, it is easy to prove that the change of the Morse index is $\eta_1(\nu_k) = j\sigma$. In any case, all bifurcation points have indices of the same sign, then for the equal mass n-body problem, these bifurcations branches must be non-admissible or connect to other relative equilibria.

7.2 Spatial spectrum for the polygonal equilibrium

Theorem 30 *Define $\nu_k = \sqrt{\mu + s_k}$ for $k \in \{1, ..., n-1\}$, and $\nu_n = \sqrt{\mu + n}$. Then, the polygonal equilibrium has a global bifurcation of periodic solutions from $2\pi/\nu_k$ with symmetries $\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2$ for each $k \in \{1, ..., n\}$.*

Proof. For $k \in \{1, ..., n-1\}$, we have that $m_{1k}(\nu) = \nu^2 - (\mu + s_k)$ changes only at ν_k. Since $\sigma = 1$, then $\eta_1(\nu_k) = 1$. For $k = n$, we have that

$$\det m_{1n}(\nu) = \nu^2\mu\left(\nu^2 - (\mu + n)\right)$$

changes only at ν_n. Since $\det m_{1n}$ is negative for $\nu < \nu_n$, then $n_{1n}(\nu_n - \rho) = 1$. Moreover, since $n_{1n}(\infty) = 0$, then $\eta_{1n}(\nu_n) = 1$. The existence of the bifurcations follows from the bifurcation theorem. ■

Since the spatial bifurcations for the polygonal equilibrium have index $\eta = 1$, then all the bifurcation branches must be non-admissible or go to the other equilibria.

7.3 Planar spectrum for the polygonal equilibrium

In the polygonal equilibrium $\bar{a}$, the frequency is $\omega = \mu + s_1$. The equations have physical meaning for positive mass, $\mu > 0$, although, the problem is well defined for $\mu > -s_1$. In this section we will analyze the spectrum of the blocks $m_{0k}(\nu)$, for $\mu > -s_1$.

Normalize the period of $m_{0k}(\nu)$ by $\sqrt{\omega}$ and define $m_k(\nu) = m_{0k}(\sqrt{\omega}\nu)$, then

$$m_k(\nu) = \omega[\nu^2 I - 2\nu(iJ)] + B_k \text{ for } k \in \{2, ..., n-2, n\} \text{ and}$$
$$m_1(\nu) = \omega[\nu^2 diag(\mu, I) - 2\nu diag(\mu, iJ)] + B_1.$$

Notice that $m_k(\nu)$ is selfadjoint, so with real eigenvalues. Since $m_{n-k}(\nu) = \bar{m}_k(-\nu)$, then the matrices $m_{n-k}(\nu)$ and $m_k(-\nu)$ have the same spectrum. Thus, the Morse numbers $n_k(\nu)$ satisfy

$$n_{n-k}(\nu) = n_k(-\nu).$$

We cannot calculate explicitly the sums s_k, but we shall use that the sums s_k are positive, satisfy $s_k = s_{n-k} = s_{n+k}$, and that s_k are increasing in k for $k \in \{0, ..., n/2\}$. This last fact is proved in the appendix. The matrices B_k are given in the proposition (12). However, in order to simplify the computations, we shall restrict the analysis to the Newton case, that is with $\alpha = 2$.

7.3.1 Block $k = n$

The block B_n is $B_n = (3/2)(\mu + s_1)(I + R)$ with $R = diag(1, -1)$. Therefore,

$$\sigma = sgn(e_1^T B_n e_1) = 1,$$

for $\mu > -s_1$.

Proposition 31 *The matrix $m_n(\nu)$ changes its Morse index only at the positive value $\nu = 1$ with*

$$\eta_n(1) = 1.$$

Proof. The block $m_n(\nu)$ is

$$m_n(\nu) = \omega[\nu^2 - 2\nu(iJ) + diag(3, 0)].$$

Thus, the determinant $d_n(\nu) = \omega^2\nu^2(\nu - 1)(\nu + 1)$ is zero only at ± 1. Since $d_n(\varepsilon) < 0$, then $n_n(\varepsilon) = 1$. Moreover, since $n_n(\infty) = 0$, then $\eta_n(1) = 1$. ■

Therefore, there is a global branch of periodic solutions bifurcating from the polygonal equilibrium starting with the period 2π and with symmetries $\tilde{\mathbb{Z}}_n(n) \times \mathbb{Z}_2$.

Remark 32 *In fixed coordinates, the central body satisfies* $q_0(t) = 0$ *and the other* n *bodies satisfy* $u_{j+1}(t) = e^{ij\zeta}u_1(t)$*. Now, if* ν *remains* 1 *on the branch, that is, if the frequency, of the solutions in fixed coordinates, is* ω*, then* $q_{j+1}(t) = e^{ij\zeta}q_1(t)$*. Thus, this bifurcation branch may be made of solutions moving on curves looking like ellipses.*

7.3.2 Blocks $k \in \{2, ..., n-2\}$

Define $d_k(\nu)$ to be the determinant of $m_k(\nu)$.

Proposition 33 *The determinant of* $m_k(\nu)$ *is*

$$d_k(\nu) = \omega^2\nu^4 + (2\alpha_k - \omega - s_1)\omega\nu^2 - 4\omega\gamma_k\nu + a_k + \mu b_k,$$

where

$$a_k = (s_1 + \alpha_k)^2 - \beta_k^2 - \gamma_k^2 \text{ and } b_k = 3(s_1 + \alpha_k + \beta_k) > 0.$$

Proof. The block $m_k(\nu)$ is

$$m_k(\nu) = (\omega\nu^2 + s_1 + \alpha_k)I + (3\mu/2)(I + R) - \beta_k R - (2\nu\omega + \gamma_k)(iJ).$$

Therefore, the determinant is

$$\begin{aligned} d_k(\nu) &= (\omega\nu^2 + s_1 + \alpha_k - \beta_k + 3\mu)(\omega\nu^2 + s_1 + \alpha_k + \beta_k) \\ &\quad - (2\omega\nu + \gamma_k)^2. \end{aligned}$$

Since $\omega = \mu + s_1$ and $d_k(0) = b_k\mu + a_k$, then

$$d_k(\nu) = \omega^2\nu^4 + (2\alpha_k - \omega - s_1)\omega\nu^2 - 4\omega\gamma_k\nu + a_k + \mu b_k.$$

■

We write the determinant in terms of ω as

$$d_k(\nu) = a\omega^2 + b\omega - c,$$

where

$$a = \nu^2(\nu^2 - 1),\ b = \nu^2(2\alpha_k - s_1) - 4\nu\gamma_k + b_k \text{ and } c = s_1 b_k - a_k.$$

As a consequence of the next lemma, we shall prove that $b^2 + 4ac$ is positive. Then $d_k(\mu, \nu)$ is zero at exactly the two real solutions

$$\omega_\pm(\nu) = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Lemma 34 *For any* $\nu \in \mathbb{R}$*, we have that* c *and* $b(\nu)$ *are positive and satisfy*

$$b^2(\nu) > 9c.$$

Proof. From the definitions of a_k and b_k we have

$$4s_1 b_k = 3s_1 s_{k-1} + 3s_1 s_{k+1} - 6s_1^2 + 18s_1 s_k,$$

and

$$4a_k = 2s_1 s_{k-1} + 2s_1 s_{k+1} - 5s_1^2 - 9s_k^2 + 18s_1 s_k + s_{k-1}s_{k+1}.$$

Therefore,

$$4c = 4(s_1 b_k - a_k) = 9s_k^2 - (s_{k-1} - s_1)(s_{k+1} - s_1).$$

Using the inequality of the appendix

$$(s_{k+1} - s_1) < 2s_k - (s_{k-1} - s_1) < 2s_k,$$

we conclude that

$$7s_k^2 < 4c \leq 9s_k^2.$$

Since $b'(\nu) = 2\nu(2\alpha_k - s_1) - 4\gamma_k$, then $b'(\nu) = 0$ at $\nu_0 = 2\gamma_k/(2\alpha_k - s_1)$. Since $2\alpha_k - s_1 > 0$, then ν_0 is a minimum of $b(\nu)$. Moreover, since $(2\alpha_k - s_1) - 2\gamma_k = s_{k-1} - s_1 \geq 0$, then $\nu_0 \in (0,1)$. Thus,

$$b(\nu_0) = \nu_0(2\gamma_k - 4\gamma_k) + b_k > b_k - 2\gamma_k.$$

Therefore,

$$4b(\nu_0) > 4b_k - 8\gamma_k = 18s_k - 6s_1 + 5s_{k-1} + s_{k+1} > 18s_k.$$

From these inequalities one has that $b^2(\nu) > 9(9s_k^2/4) > 9c$. ■

For $\nu^2 > 1$, since a is positive, then $b^2 + 4ac$ is positive. For $\nu^2 \in (0,1)$, since $a \in (-1/4, 0)$ and $b^2(\nu) > 4c$, then $b^2 + 4ac \geq 4c(1+a) \geq 0$. Consequently, the two solutions $\omega_\pm(\nu)$ are real for $\nu \neq -1, 0, 1$.

Remember that the equations of the bodies have meaning only for $\omega = \mu + s_1 > 0$. For $\nu^2 > 1$, since $4ac$ is positive, then $\omega_+(\nu)$ is positive and $\omega_-(\nu)$ is negative. For $\nu^2 \in (0,1)$, since a and $4ac$ are negative, then the two roots $\omega_\pm(\nu)$ have the sign of b. Since b is positive, then the two solutions $\omega_\pm(\nu)$ are positive for $\nu^2 \in (0,1)$.

Defining $\mu_\pm(\nu) = \omega_\pm(\nu) - s_1$, the matrix $m_k(\mu, \nu)$ changes its Morse index only at the two curves

$$\mu_+(\nu) \text{ for } \nu \in \mathbb{R}, \text{ and } \mu_-(\nu) \text{ for } \nu \in (-1,0) \cup (0,1).$$

These curves satisfy the inequalities $-s_1 < \mu_+(\nu) < \mu_-(\nu)$ for $|\nu| \in (0,1)$. Moreover, since $a \to 0$ when $|\nu| \to \{0,1\}$, then $\mu_-(\nu) \to \infty$ when $|\nu| \to \{0,1\}$. Furthermore, since $\omega_+(\nu) \to 0$ when $|\nu| \to \infty$, then $\mu_+(\nu) \to -s_1$ when $|\nu| \to \infty$.

Definition 35 *Let m_0 be the maximum of $\mu_+(\nu)$ in $\mathbb{R}$, m_+ be the minimum of $\mu_-(\nu)$ on $(0,1)$, and m_- be the minimum of $\mu_-(\nu)$ on $(-1,0)$.*

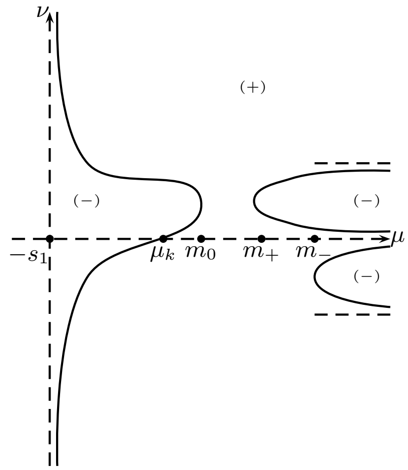


Figure 4: Graph of $d_k(\mu, \nu) = 0$.

Proposition 36 *For $k \in \{2, ..., n-2\}$, if $\mu \in (-s_1, \mu_k)$, then $m_k(\nu)$ changes its Morse index at a positive value ν_k, with*

$$\eta_k(\nu_k) = 1.$$

For $k \in \{2, ..., [n/2]\}$, if $\mu \in (\mu_k, m_0)$, then $m_k(\nu)$ changes its Morse index at two positive values $\bar{\nu}_k < \nu_k$, with

$$\eta_k(\bar{\nu}_k) = -1 \text{ and } \eta_k(\nu_k) = 1.$$

For $k \in \{2, ..., n-2\}$, if $\mu \in (m_+, \infty)$, then $m_k(\nu)$ changes its Morse index at two positive values $\bar{\nu}_k < \nu_k < 1$, with

$$\eta_k(\bar{\nu}_k) = -1 \text{ and } \eta_k(\nu_k) = 1.$$

Proof. Since $\mu_k = -b_k/a_k$ is the only zero of $d_k(\mu, 0)$, then $\mu_k \leq m_0$. Notice that the function $d_k(\nu) + 4\omega\gamma_k\nu$ is even in ν, then

$$d_k(\mu, \nu) = d_k(\mu, -\nu) - 8\omega\gamma_k\nu.$$

Since $\gamma_k > 0$ for $k \in [2, ..., n/2) \cap \mathbb{N}$, then $d_k(\mu, \nu) < d_k(\mu, -\nu)$ for $\nu \in \mathbb{R}^+$. Thus, the maximum m_0 is reached for some $\nu > 0$, and $m_+ < m_-$.

Since $m_k(\nu)$ is a 2×2 matrix, then the Morse number is $n_k(\nu) = 1$ when $d_k(\nu)$ is negative. Hence, the Morse index is $n_k(\mu, \nu) = 1$ in Ω, where

$$\Omega = \{(\mu, \nu) : \mu < \mu_0(\nu), \mu > \mu_\pm(\nu)\}.$$

Since $n_k(\nu) = 0$ for ν large enough, and since $m_k(\nu)$ changes its Morse index only at $\partial\Omega$, then $n_1(\mu, \nu) = 0$ in $\bar{\Omega}^c$.

As we have seen before, $\sigma = 1$ for $\mu > -s_1$, then, for the first case, we have that $\eta_k(\nu_k) = 1 - 0$. For the other two cases, there are two values, $\bar{\nu}_k < \nu_k$, with $\eta_k(\bar{\nu}_k) = 0 - 1$ and $\eta_k(\nu_k) = 1 - 0$. ■

Remark 37 *Since*

$$d_k''(\nu) = 2(6\omega^2\nu^2 + (2\alpha_k - \mu - 2s_1)\omega),$$

then $d_k''(\nu)$ *is positive for* $\mu < 2(\alpha_k - s_1)$. *Thus, for* $\mu \in (-s_1, 2(\alpha_k - s_1))$ *the matrix* $m_k(\nu)$ *changes its Morse index only at* ν_k, *for the first case. Now, since* $d_k(\nu)$ *is a polynomial in* ν *of degree* 4, *then* $d_k(\nu)$ *has at most four zeros. This is the case for* $\mu > \max\{m_+, m_-\}$, *and then the two positive solutions for the third case are the only ones.*

From the bifurcation theorem we have the following result, for $n \geq 4$.

Theorem 38 *The polygonal equilibrium has a global bifurcation of planar periodic solutions with symmetries* $\tilde{\mathbb{Z}}_n(k)$ *for each* $k \in \{2, ..., n-2\}$ *and* $\mu \in (-s_1, \mu_k)$. *When* $\mu \in (-s_1, 2(\alpha_k - s_1))$, *these bifurcation branches are non-admissible or go to another equilibrium .*

For each $k \in \{2, ..., [n/2]\}$ *and* $\mu \in (\mu_k, m_0)$, *and for each* $k \in \{2, ..., n-2\}$ *and* $\mu \in (m_+, \infty)$, *the polygonal equilibrium has two global bifurcations of planar periodic solutions with symmetries* $\tilde{\mathbb{Z}}_n(k)$.

Remark 39 *If we had proved that there are only three solutions for* $d_k(\nu) = 0$ *and* $d_k'(\nu) = 0$, *then these solutions would have been* m_0 *and* $m_\pm$. *In that case, the only points where the Morse index changes are those of the previous theorem .*

We can prove this result for $k = n/2$, *and large* n, *since* $\gamma_{n/2} = 0$. *In fact, one has that* $m_+ = m_-$, *and* $m_0 = \mu_{n/2}$ *if* n *is large enough. Since* $\gamma_{n/2} = 0$, *the graph for* $d_{n/2}$ *is symmetric with respect to the* μ*-axis and one has two curves for* μ *as a function of* ν.

In order to compute $m_+ = m_-$, *notice that*

$$d_k'(\nu) = 2\nu\omega^2(2\nu^2 + \omega^{-1}(2\alpha_{n/2} - 2s_1 - \mu)) = 0.$$

Substituting $\nu^2 = -\omega^{-1}(2\alpha_{n/2} - 2s_1 - \mu)/2$ *in* $d_k(\nu) = 0$, *we obtain*

$$(2\alpha_{n/2} - 2s_1 - \mu)^2 - 4(a_{n/2} + \mu b_{n/2}) = 0.$$

From this quadratic equation, we get that $m_+ = b + \sqrt{b^2 + c}$, *with*

$$b = 2(b_{n/2} + \alpha_{n/2} - s_1) \text{ and } c = 4(a_{n/2} - (\alpha_{n/2} - s_1)^2).$$

The other root, $\tilde{m} = b - \sqrt{b + c}$ *will be valid provided the expression for* ν^2 *is positive, giving at most two values for* μ, *and* $\nu > 0$, *where the derivative of* μ *with respect to* ν *is* $\mu' = 0$. *At these points one has that*

$$\mu'' = -d_{\nu\nu}/d_\omega = 8\omega(2\alpha_{n/2} - 2s_1 - \mu)/(b - \mu).$$

In particular, $\mu'' > 0$ *at* m_+ *and* $\mu'' < 0$ *at* $\tilde{m}$.

It is easy to see that $\tilde{m} < 2(\alpha_{n/2} - s_1)$ *(and there the corresponding* ν^2 *is negative) if and only if* $\mu_{n/2}$ *satisfies the same inequality and, in this case,* $\mu_{n/2}$ *is the only maximun of this piece of the graph.*

For n *large, the asymptotics of the appendix show that* $\mu_{n/2}$ *behaves like* $\sigma n^3/3$, *while* $2(\alpha_{n/2} - s_1)$ *is like* σn^3, *which implies that, for* n *large,* $m_0 = \mu_{n/2}$.

7.3.3 Blocks $k \in \{1, n-1\}$

From the definition, the block $m_1(\nu)$ is

$$\begin{pmatrix} \mu\left(\omega\nu^2 - 2\nu\omega + s_1 + \mu + n/2\right) & -2\,(n/2)^{1/2}\,\mu & -\,(n/2)^{1/2}\,\mu i \\ -2\,(n/2)^{1/2}\,\mu & \omega\nu^2 + s_1 + \alpha_1 + 3\mu & \alpha_1 i + 2\nu\omega i \\ (n/2)^{1/2}\,\mu i & -\alpha_1 i - 2\nu\omega i & \omega\nu^2 + s_1 + \alpha_1 \end{pmatrix}.$$

Let us define a_1 and b_1 as

$$a_1 = (2s_1 + n)(2s_1 + s_2)/4 \text{ and } b_1 = 3(4s_1 + s_2 - 2n)/4.$$

The determinant of $m_1(\nu)$ may be written as

$$\det m_1(\nu) = \omega\mu(\nu - 1)^2 d_1(\nu),$$

where $d_1(\nu)$ is the polynomial

$$d_1(\nu) = \nu^2(\nu^2 - 1)\omega^2 + \left[(s_2 - 2s_1 + n)\,\nu^2/2 - (s_2 - n)\nu\right]\omega + (a_1 + \mu b_1).$$

Therefore, we need to find the points where $d_1(\nu)$ is zero.

Lemma 40 *The functions b_1, $2\alpha_1 - s_1$, $s_1 - n$, $s_2 - n$ and $8(s_1 - n) + 9s_2$ are positive, at least for $n > 1071$. For $n \geq 3$, we get numerically that $s_1 - n$ is negative if $n \leq 472$ and positive if $n \geq 473$, that $s_2 - n$ is negative if $n \leq 11$ and positive if $n \geq 12$, and*

$$sgn(b_1) = sgn(2\alpha_1 - s_1) = sgn(8(s_1 - n) + 9s_2) = \begin{cases} -1 \text{ for } n \leq 6 \\ +1 \text{ for } n \geq 7 \end{cases}.$$

Proof. Since $\sin x < x$, then

$$s_1 = \frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{\sin j\zeta/2} \geq \frac{1}{2}\sum_{j\in[1,n/2)\cap\mathbb{N}}\frac{1}{j\zeta/2} \geq \frac{1}{\zeta}\ln(n/2) = n\frac{\ln(n/2)}{2\pi}.$$

Since $\ln(n/2) \geq 2\pi$ for $n > 1071$, then $s_1 > n$ at least for $n > 1071$.

Using that $s_2 > s_1$ and $b_1 = 4(2(s_1 - n) + s_2 + 2s_1)/4$, we conclude that b_1, $s_2 - n$ and $8(s_1 - n) + 9s_2$ are positive for $n > 1071$. Finally, from proposition (57), we have that $s_2 = 4s_1 - \bar{s}_1$ with $\bar{s}_1 = \sum_{j=1}^{n-1}\sin(j\zeta/2) < n$, then,

$$2\alpha_1 - s_1 = s_2/2 - s_1 = s_1 - \bar{s}_1/2 > s_1 - n \geq 0$$

for $n > 1071$. ■

We shall analyze the graph of $d_1(\nu) = 0$ only for positive masses, $\mu > 0$.

Proposition 41 *For $n \geq 7$, the polynomial $d_1(\mu, \nu)$ is zero only for the function*

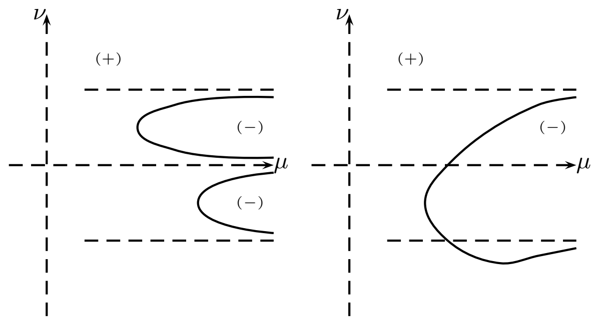
$$\mu_0(\nu) : (-1, 0) \cup (0, 1) \to \mathbb{R}^+,$$

where $\mu_0(\nu) \to \infty$ when $|\nu| \to 0, 1$.

For $n \in \{3, 4, 5, 6\}$, the polynomial $d_1(\mu, \nu)$ is zero only on the curve

$$(\mu_0, \nu_0) : \mathbb{R} \to \mathbb{R}^+ \times (-\infty, 1),$$

where $\mu_0(t) \to \infty$ and $\nu_0(t) \to \pm 1$ when $t \to \pm\infty$.



(a) $n > 6$ (b) $n = 3, 4, 5, 6$

Figure 5: Graph of $d_1(\mu, \nu) = 0$.

Proof. Since $\omega = \mu + s_1$,

$$d_1(\mu, \nu) = \nu^2 \left(\nu^2 - 1\right) \mu^2 + [s_1 b_1 - a_1 + c_1 + s_1^2 \left(\nu^4 - \nu^2\right)]\mu/s_1 + c_1,$$

where $s_1 b_1 - a_1 = (8s_1 + s_2)(s_1 - n)/4$ and

$$c_1(\nu) = [2s_1(\nu - 1)^2 + n][2s_1(\nu + 1)^2 + s_2]/4.$$

For each fixed ν, set $d_1(\nu)$ be the value of the determinant. Then, its derivative $d_1'(\mu)$ is zero at the only critical point μ_c such that

$$\frac{1}{s_1}(s_1 b_1 - a_1 + c_1 + s_1^2 \left(\nu^4 - \nu^2\right))\mu_c = -2\nu^2 \left(\nu^2 - 1\right) \mu_c^2.$$

For $|\nu| < 1$, since $d_1(\mu_c, \nu) = -\nu^2 \left(\nu^2 - 1\right) \mu_c^2 + c_1$ is positive, then the polynomial $d_1(\mu)$ has at most one zero for $\mu > \mu_c$. Since $d_1/\omega^2 \to \nu^2(\nu^2 - 1)$ when $\mu \to \infty$, and $d_1(0, \nu) = c_1$ is positive, then the polynomial d_1 has exactly one zero at $\mu_0(\nu) > 0$ for each $|\nu| \in (0, 1)$.

In order to obtain the limits of the function $\mu_0(\nu)$, when $|\nu| \to 0, 1$, we need to study the zeros of $d_1(\mu, \nu)$ for $|\nu| = 0, 1$. For $\nu = 0$, the polynomial $d_1(\mu, 0)$ is zero at $\mu_0 = -a_1/b_1$. Since a_1 is positive and b_1 is negative for $n \leq 6$ only, then μ_0 is positive for $n \leq 6$ only. For $\nu = 1$, the polynomial $d_1(\mu, 1)$ is zero at the point $\mu_{+1} = -n$. For $\nu = -1$, the polynomial $d_1(\mu, -1)$ is zero at

$$\mu_{-1} = -s_2(8s_1 + n)/(8s_1 - 8n + 9s_2),$$

which is positive for $n \leq 6$ only.

Hence, for $n \geq 7$, the polynomial $d_1(\mu, \nu)$ is positive at $|\nu| = 0, 1$, for any $\mu \geq 0$. Therefore, the functions $\mu_0(\nu) \to \infty$ when $|\nu| \to 0, 1$. Moreover, since $c_1(\nu)$ is increasing for $\nu > 1$, then $d_1(\nu) \geq d_1(1) > 0$, for $\nu > 1$. Similarly, one has that $d_1(\nu) > 0$, for $\nu < -1$. We conclude that the only zeros of $d_1(\mu, \nu)$ for $\mu \geq 0$ are for the function $\mu_0(\nu)$.

For $n \leq 6$, we may prove, in a similar way, that $d_1(\mu,\nu)$ is positive, for $\nu > 1$. As a consequence, the function $\mu_0(\nu) \to \infty$ when $\nu \to 1$. Also, the function $\mu_0(\nu)$ has a continuous extension, to $\nu = 0, -1$, by $\mu_0(0) = \mu_0$ and $\mu_0(-1) = \mu_{-1}$. Furthermore, the continuum of zeros of $d_1(\mu,\nu)$ at $(\mu_{-1}, -1)$ crosses the line $(\mu, -1)$ only once, then this continuum must go to $(\mu,\nu) = (\infty, -1)$. This statement follows from the fact that $d_1(\mu,\nu)$ is positive for $\nu < -1$ and $\mu = 0$, for $\mu > 0$ and $|\nu|$ large, and for $\nu < -1$ and μ large. ∎

Remark 42 *The equation $d_1(\mu,\nu) = 0$ is quadratic in μ, thus we can find explicitly two roots $\mu_\pm(\nu)$of $d_1(\mu,\nu) = 0$. In principle, we could analyze these roots analogously to the case $k \in \{2, ..., n-2\}$. However, there are many cases depending on n. The main case is for $n \geq 473$, where we may prove that the graph of $d_1(\mu,\nu) = 0$ looks like the graph of $d_k(\mu,\nu) = 0$, for $\mu > -s_1$.*

Definition 43 *For $n \geq 7$, define m_+ to be the minimum of the function $\mu_0(\nu)$ in $(0,1)$, and m_- the minimum in $(-1,0)$. For $n \in \{3, ..., 6\}$ define m_0 to be the minimum of $\mu_0(t)$ for $t \in \mathbb{R}$.*

Proposition 44 *For masses $\mu > 0$ we have the following cases:*

(a) *For $n \in \{3, ..., 6\}$, the matrix $m_1(\nu)$ changes its Morse index at the two values $\nu_- < \bar{\nu}_- < 0$ if $\mu \in (m_0, \mu_1)$, and at $\nu_- < 0 < \nu_+ < 1$ if $\mu \in (\mu_1, \infty)$.*

(b) *For $n \geq 12$, the matrix $m_1(\nu)$ changes its Morse index at $0 < \bar{\nu}_+ < \nu_+ < 1$ if $\mu \in (m_+, m_-)$, and at the four values of ν*

$$-1 < \nu_- < \bar{\nu}_- < 0 < \bar{\nu}_+ < \nu_+ < 1$$

if $\mu \in (m_-, \infty)$. For $n \in \{7, ..., 11\}$, this statement is true with m_+ and m_- interchanged.

In both cases, we have that

$$\eta_1(\bar{\nu}_\pm) = -1 \text{ and } \eta_1(\nu_\pm) = 1.$$

Proof. For $n \geq 12$, using $s_2 > n$, we may prove that $c_1(\nu) < c_1(-\nu)$. Then, $d_1(\mu,\nu) < d_1(\mu,-\nu)$, for $\nu > 0$. Therefore, $m_+ < m_-$, for $n \geq 12$. In a similar way, we may prove that $m_- > m_+$, for $7 \leq n \leq 11$.

The trace of $m_1(\nu)$ is

$$T_1 = \mu\omega(\nu-1)^2 + 2\omega(\nu^2+1) + \mu(n/2+1) + 2\alpha_1.$$

Since T_1 is positive for $\mu > 0$, then the Morse index satisfies $n_1(\nu) = 1$, when $\det m_1(\nu)$ is negative, and $n_1 \in \{0, 2\}$, when $\det m_1(\nu)$ is positive.

Let Ω be the connected component where $d_1(\mu,\nu)$ is negative, for $n \geq 7$ this set is $\Omega = \{(\mu,\nu) : \mu > \mu_0(\nu)\}$. From the definition, we have that $n_1 = 1$ in $\Omega - \{(\mu, \pm 1)\}$. Now, for $\mu = 0$, the matrix $m_1(0,\nu)$ has eigenvalues 0, $s_1(\nu-1)^2$

and $s_1(\nu+1)^2+2\alpha_1$, hence $n_1(\varepsilon,\nu)\le 1$, for small ε. Since $n_1\in\{0,2\}$ and $n_1\le 1$ in $\bar\Omega^c-\{(\mu,\pm 1)\}$, then $n_1=0$ there. Thus, we have that $\eta_1(\pm 1)=0-0$, $\eta_1(\bar\nu_\pm)=0-1$ and $\eta_1(\nu_\pm)=1-0$. ■

From the bifurcation theorem, we get the following result:

Theorem 45 *For $n\ge 12$, if $\mu>m_+$, the polygonal equilibrium has two global bifurcation branches of planar periodic solutions with symmetries $\tilde{\mathbb{Z}}_n(1)$. If $\mu>m_-$, there are two more bifurcation branches with symmetries $\tilde{\mathbb{Z}}_n(n-1)$. For $n\in\{7,...,11\}$, the same statement is true interchanging m_+ and m_-.*

For $n\in\{3,...,6\}$, if $\mu>\mu_1$, the polygonal equilibrium has one global bifurcation branch of planar periodic solutions with symmetries $\tilde{\mathbb{Z}}_n(1)$, and one more with symmetries $\tilde{\mathbb{Z}}_n(n-1)$. If $\mu\in(m_0,\mu_1)$, there are two global bifurcation branches of planar periodic solutions with symmetries $\tilde{\mathbb{Z}}_n(n-1)$.

Remark 46 *When $\mu=0$, the central body does not exist, then the block $m_1(\nu)$ is*

$$m_1(\nu)=\begin{pmatrix} s_1\nu^2+s_1+\alpha_1 & \alpha_1 i+2\nu s_1 i\\ -\alpha_1 i-2\nu s_1 i & s_1\nu^2+s_1+\alpha_1\end{pmatrix}.$$

In this case $m_1(\nu)$ has eigenvalues $s_1(\nu-1)^2$ and $s_1(\nu+1)^2+2\alpha_1$. Then $n_1(\nu)=0$ for all ν, and there is no bifurcation in this case.

8 Remarks

8.1 The case $n=2$

The irreducible representations for $n=2$ are different from those for larger n, due to the action of $\mathbb{Z}_2$. This fact gives a change in the planar representation for $k=1$, but the change of variables for $k=2$ remains the same as for larger n's. In particular, the bifurcation results for the spatial case and for the planar case, with $k=2$ are similar to the ones already given.

For $k=1$, define the isomorphism $T_1:\mathbb{C}^4\to W_1$ as

$$T_1(v,w)=(v,2^{-1/2}w,2^{-1/2}w)\text{ with}$$
$$W_1=\{(v,w,w):v,w\in\mathbb{C}^2$$

It is not difficult to find the action of $\tilde{\mathbb{Z}}_2$ on W_1 and to compute the Hessian of the potential V. The matrix B_2 is the same as before, but B_1 is now a 4×4 complex matrix. For $\alpha=2$, one has that $s_1=1/4$ and $\omega=\mu+1/4$. The matrix $m_1(\nu)$ is

$$\begin{pmatrix} \mu\left(\omega\nu^2+\mu+17/4\right) & -2i\nu\omega\mu & -2\,(2)^{1/2}\mu & 0\\ 2i\nu\omega\mu & \mu\left(\omega\nu^2+\mu-7/4\right) & 0 & (2)\,1/2\mu\\ -2\,(n/2)^{1/2}\mu & 0 & \omega\nu^2+3\mu+1/4 & -2i\nu\omega\\ 0 & (2)^{1/2}\mu & 2i\nu\omega & \omega\nu^2+1/4\end{pmatrix}.$$

The determinant of $m_1(\nu)$ may be written as

$$\det m_1(\nu) = 2^{-8}\mu^2(4\mu+1)^2(\nu^2-1)^2 d_1(\nu),$$

where $d_1(\nu)$ is the polynomial

$$d_1(\nu) = \left(16\mu^2+8\mu+1\right)\nu^4 + \left(-16\mu^2+20\mu+6\right)\nu^2 - (84\mu+119).$$

For $\mu > 0$, $d_1(\nu)$ is zero only for $\pm\nu_1$, where ν_1 is the positive solution of

$$\nu_1^2 = \left(2\mu - 3 + 2\left(\mu^2+18\mu+32\right)^{1/2}\right)/(4\mu+1).$$

Since $\nu_1 > 1$ for $\mu > 0$, it is enough to know the eigenvalues of $m_1(\nu)$ for one value of μ, for instance, $\mu = 1$. These eigenvalues are $5(\nu \pm 1)^2$ and $\left(5\nu^2 \pm 2(25\nu^2+81)^{1/2}+11\right)/4$.

Thus, one sees that $n_1 = 1$ for $-\nu_1 < \nu < \nu_1$ and $n_1 = 0$ otherwise. Hence, $\eta_1(\nu_1) = 1$ and $\eta_1(1) = 0$. Therefore,

Theorem 47 *There is a global bifurcation branch of planar periodic solutions, from $2\pi/\nu_1$, with symmetries $\tilde{\mathbb{Z}}_2(1)$. There is also a global bifurcation of planar periodic solutions, from the period 2π, and symmetry $\tilde{\mathbb{Z}}_2(2)\times\mathbb{Z}_2$, that is*

$$u_0(t) = 0 \text{ and } u_2(t) = -u_1(t).$$

Finally, there are two spatial bifurcation branches, from $\nu = \sqrt{\mu+1/4}$ and $\nu = \sqrt{\mu+2}$, with symmetries $\tilde{\mathbb{Z}}_2(1)\times\mathbb{Z}_2$ and $\tilde{\mathbb{Z}}_2(2)\times\mathbb{Z}_2$, respectively, that is, the first type is such that $u_0(t) = 0$ and $z_0(t) = 0$,

$$u_2(t) = -u_1(t) \text{ and } z_2(t) = -z_1(t),$$

while, the second type is with $u_0(t) = 0$ and $z_0(t+\pi) = z_0(t)$,

$$u_2(t) = -u_1(t) = -u_1(t+\pi) \text{ and } z_2(t) = z_1(t) = -z_1(t+\pi).$$

8.2 Resonances

We have proved that there is a bifurcation of periodic solutions with the symmetries (7) when the block $M_1(\nu)$ changes its Morse index at ν_1. However, we cannot deduce, from the symmetries (7), that the solutions are truly spatial, because all the spatial coordinates can be zero: $z_j(t) = 0$. In fact, there is a non-resonance condition to assure that the solutions are truly spatial.

Proposition 48 *If $M_1(\nu)$ changes its Morse index at ν_1, and the matrix $M_0(2l\nu_1)$ is invertible for all $l \in \mathbb{N}$, then the equilibrium x_0 has a bifurcation of truly spatial periodic solutions from ν_1 with symmetries (7) .*

Proof. We need to prove that near $(\bar{a}, \nu_k)$ there are no planar solutions with symmetry $\tilde{\mathbb{Z}}_2$, i.e.

$$x_j(t) = x_j(t+\pi),\ y_j(t) = y_j(t+\pi) \text{ and } z_j(t) = 0.$$

These solutions have isotropy group $\mathbb{Z}_2 \times \mathbb{Z}_2$, one $\mathbb{Z}_2$ is generated by κ and the other by $\pi \in S^1$.

Let $f'(\bar{a})$ be the derivative of the bifurcation operator $f(x)$, at $\bar{a}$, and define $f'(\bar{a})|_{\mathcal{W}^H}$ as the derivative in the fixed point space of $H = \mathbb{Z}_2 \times \mathbb{Z}_2$. The linear map $f'(\bar{a})|_{\mathcal{W}^H}$ has a block of the form $M_0(2l\nu)$, for each Fourier mode l.

By assumption, the blocks $M_0(2l\nu_1)$ are invertible, then the derivative $f'(\bar{a})|_{\mathcal{W}^H}$ is invertible. Using the implicit function theorem, we may conclude the non-existence of periodic solutions with isotropy group $\mathbb{Z}_2 \times \mathbb{Z}_2$ near $(\bar{a}, \nu_1)$. As a consequence, the bifurcation from ν_1 must be spatial. ■

For the polygonal equilibrium, we have proved before that the block $m_{1k}(\nu_k)$ changes its Morse index at $\nu_k = \sqrt{\mu + s_k}$. Restricting $f'(\bar{a})$ to the group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we may prove, analogously to the general case, that the bifurcation from ν_k is truly spatial if $m_{0k}(2l\nu_k)$ is invertible for all l's.

For $k \in \{1, ..., n-1\}$, we have proved that the block $m_{0k}(\nu)$ is invertible for $|\nu| > \sqrt{\mu + s_1}$, when $\mu > m_0$: $\nu = 1$ in the previous graph corresponds to $\nu = \sqrt{\mu + s_1}$ before the normalization used in the study of $m_k(\nu)$. Since $2l\nu_k > \sqrt{\mu + s_1}$ for all l's, then $m_{0k}(2l\nu_k)$ is invertible for all l's, when $\mu > m_0$. Therefore, the bifurcation from ν_k is truly spatial for $k \in \{1, ..., n-1\}$, when $\mu > m_0$.

If one uses the same normalization done for the planar spectrum, that is $m_{0k}(\sqrt{\omega}\nu)$, for the spatial spectrum, then one has that, for $k = 1, ..., n-1$,

$$m_{1k}(\sqrt{\omega}\nu) = \nu^2(\mu + s_1) - (\mu + s_k),$$

and the corresponding expression for $k = n$.

Then, the block m_{1k} changes its Morse index at $\nu_k = \sqrt{(\mu + s_k)/(\mu + s_1)}$. Restricting $f'(\bar{a})$ to the group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we may prove, analogously to the general case, that the bifurcation from ν_k is truly spatial if $m_{0k}(2l\nu_k)$ is invertible for all l's.

For $k \in \{1, ..., n-1\}$, we have proved that the block $m_{0k}(\nu)$ is invertible for $|\nu| > 1$, when $\mu > m_0$. Since $2l\nu_k > 1$ for all l's, then $m_{0k}(2l\nu_k)$ is invertible for all l's, when $\mu > m_0$. Therefore, the bifurcation from ν_k is truly spatial for $k \in \{1, ..., n-1\}$, when $\mu > m_0$. For other values of $\mu < m_0$, one has that $\nu_k^2 = (\mu + s_k)/(\mu + s_1)$, and $\omega = \mu + s_1 = (s_k - s_1)/({\nu_k}^2 - 1)$. The equation $d_k(2l\nu_k) = 0$ leads to finding the zeros of a 4th degree polynomial in ν_k, with coefficients which depend on l. For the piece of the graph where $\nu_0 < 1$, one cannot have the resonance $\nu_0 = 2l\nu_k$ and, for the rest of the graph which is asymptotic to the line $\mu = -s_1$, there will be only a finite number of l's, where one could have the equality of the frequencies. As a consequence, there is only a finite number of possible resonances, for $k = 1, ..., n-1$.

For $k = n$, ${\nu_n}^2 = (\mu + n)/(\mu + s_1)$ and $\nu_0 = 1$, so that there is at most a finite number of l's and positive μ's which satisfy the relation $\nu_0 = 2l\nu_n$.

Remark 49 *If one has a resonance between spatial frequencies, for k_1 and k_2, such that $\nu_2 = l\nu_1$, then one has the relation $\mu(l^2 - 1) = s_{k_2} - l^2 s_{k_1}$, where s_{k_2} is replaced by n for $k_2 = n$. If $l = 1$ and $k_2 \neq n$, then $k_2 = n - k_1$, and the bifurcating orbit for k_2 is the one for k_1 taken in the opposite direction, due to the symmetry $\tilde{\kappa}$.*

If $l = 1$ and $k_2 = n$, then one needs $s_{k_1} = n$ for a $1-1$ resonance. We know that $s_{n/2} > ... > s_1$, and we have seen, in the last lemma , that $s_1 > n$ for $n > 472$, that $s_2 > n > s_1$ for $12 \leq n \leq 472$ and $n > s_2$ for $3 \leq n \leq 11$. Furthermore, numerically one computes $n > s_3$ for $6 \leq n \leq 7$ and $s_3 > n$ for $8 \leq n \leq 11$. Since all these inequalities between $s_0 = n$ and s_k are strict, then we may conclude that there are strict inequalities between ν_0 and ν_{k_1}, that is there is no $1-1$ resonance in this case.

For $k_2 < n$, we may assume that $1 \leq k_1 < k_2 \leq n/2$ and we need, for $\mu \geq 0$, to have $l^2 \leq s_{k_2}/s_{k_1}$. Using the recurrence formulae of the appendix, we see that $s_{k_2}/s_{k_1} < k_2^2 - k_1^2 + 1 \leq k_2^2$. Hence, the number of possible subharmonic resonances is finite, as well as the number of possible solutions to the relation $\mu = (s_{k_2} - l^2 s_{k_1})/(l^2 - 1)$.

Remark 50 *For resonances between planar frequencies, if $\nu_2 = l\nu_1$ and $d_{k_j}(\nu_j) = 0$ for $j = 1, 2$, it is easy to see that ν on the graph of $d_k(\nu) = 0$ is bounded above, for $\mu \geq 0$ and away from 0 for all positive bounded μ's, except for small neighborhoods of μ_k. Thus, the number of possible l's is finite. Now, if $k_j < n$, $d_{k_2}(l\nu_1) - l^2 d_{k_1}(\nu_1)$ is a polynomial of degree 2 in μ and ν_1, which can be solved for μ in terms of ν_1 and $d_{k_1}(\nu_1)$ gives a polynomial of degree 16 in ν_1, if $l > 1$, and of degree 4 if $l = 1$. In any case, we obtain at most 16 solutions for ν_1 and 32 for μ, i.e., a finite number of possible resonances, with l bounded if μ is positive, bounded, and not in a neighborhood of μ_{k_1}, where ν_1 is small.*

Note that, if $k_2 = n$, then $\nu_2 = 1$ and there is a resonance for $\nu_1 = 1/l$ and the corresponding μ, with l going to infinity as μ goes to μ_{k_1}. For $k_1 \leq n/2$, assume that the graph for $d_{k_2}(\nu)$ covers a small interval $[\mu_{k_1}, \tilde{\mu}]$. Then, if $\tilde{l}$ is such that $\tilde{l}\nu_{k_1}(\tilde{\mu}) \geq nu_{k_2}(\tilde{\mu})$, the graphs for $d_{k_1}(l\nu)$ will cut the graph for $d_{k_2}(\nu)$, for any $l \geq \tilde{l}$, in the interval, at $\mu(l)$ with the limit μ_{k_1}, giving rise to an infinite number of resonances. This situation will happen if $\mu_{k_2} > \mu_{k_1}$, which is the case for n large enough (see remark 29 in [19]). But, for μ different from μ_{k_1}, one has a finite number of possible resonances.

A similar argument may be done for $k_1 > n/2$ and intervals at the left of μ_{k_1}.

Note that, at μ_{k_1} one has a two-dimensional kernel for the linearization at the relative equilibrium (for k_1 and n) and the orbit is no longer hyperbolic, so one needs a different computation of the bifurcation index.

Remark 51 *Assume that there is no resonance, that is that $d_j(l\nu_k) \neq 0$, for $j \neq k$ and any $l \geq 1$, and $\nu_k \neq 1$. Then the equations for $j \neq k$, $j \neq n$ may be solved close to the bifurcation orbit, by an implicit function argument in terms of the component for k. The equation for n, which has a one dimensional kernel, is solved by a Poincaré section argument which is used in [24] in order*

to compute the orthogonal index for an isolated hyperbolic orbit (theorem 3.1 p. 247 in the above reference). One has then a reduction to one mode, a single k and a one-dimensional complex orthogonal equation. The non-trivial solutions of this reduced system have a least period equal to 2π, which implies that, except for a finite number of μ's, if different from μ_k, the branches which bifurcate from the relative equilibrium, for fixed μ, will be different.

8.3 Linear stability

The linearization of the equation at the equilibrium x_0, is

$$\mathcal{M}\ddot{u} + 2\sqrt{\omega}\mathcal{M}\bar{\mathcal{J}}\dot{u} = D^2V(x_0)u. \tag{12}$$

For this problem, an equilibrium is linearly stable if the linear equation (12) has $6(n+1)$ linearly independent solutions of the form $u(t) = e^{i\nu_0 t}v_0$ with $\nu_0 \in \mathbb{R}$, for $n+1$ bodies.

Now, if $\nu_0 \in \mathbb{R}$ is a bifurcation value, that is the determinant is 0 at ν_0, then there is a vector v_0 such that $M(\nu_0)v_0 = 0$. Since

$$-\mathcal{M}\ddot{u} - 2\sqrt{\omega}\mathcal{M}\bar{\mathcal{J}}\dot{u} + D^2V(x_0)u = e^{i\nu_0 t}M(\nu_0)v_0 = 0,$$

then the function $u(t) = e^{i\nu_0 t}v_0$ is a solution of the linearized equation (12).

Therefore, for each point ν_0 where $M(\nu)$ is singular, the equilibrium has a direction where the linearized equation is stable. Now, ν_0 may be a multiple zero of the characteristic determinant and one could have a polynomial increase in t. A weaker definition of stability is that of spectral stability, where the equilibrium x_0 is spectrally stable if $M(\nu)$ is non-invertible at $6(n+1)$ values of ν, counted with multiplicities.

As a consequence, we may deduce the spectral stability from the spectral analysis which we did to prove bifurcation. We shall recover, with somewhat different arguments, the results of [32], of [35] and others.

For the polygonal equilibrium, we may give a full description of stability. We have proved that each spatial block $m_{1k}(\nu)$ is non-invertible at exactly one positive value, for $k \in \{1, ..., n\}$, and $m_{1n}(0)$ has a one-dimensional kernel, with a double zero. Then, the spatial spectrum gives n stable solutions. The kernel of $m_{1n}(0)$ is generated by the vector $(1, \sqrt{n})$, which gives, under the transformation P the vector $(1, 1, , , , 1) = e$. The eigenvectors corresponding to the double root are $(a+bt)e$. In fact, since $\sum_i a_{ij} = 0$, the vector e generates the kernel of the vertical Hessian. Note that, if one sums all the vertical components of the equations, one gets $\sum \ddot{z}_j = 0$ and, if the initial conditions for $\sum z_j$ are 0, for position and speed, then this sum remains 0. Hence, by imposing these restrictions, one obtains a linear stability in the vertical direction.

Next, we analyze the planar spectrum.

Proposition 52 *Let m_* be the largest of the m_+'s, for $k = 1, ..., n$. Then, the polygonal equilibrium is spectrally stable only if $n \geq 7$ and $\mu > m_*$. It will be linearly stable if one restricts vertical translations, rotations around the center of mass and scaling of the frequency.*

Proof. For $k \in \{2, ..., n-2\}$, we have proved that the 2×2 matrix $m_k(\nu)$ is non-invertible at four values of ν, when $\mu > m_*$. For $k = n$, the 2×2 matrix $m_n(\nu)$ is non-invertible at one positive and one negative value, and 0 is a double root, with a kernel generated by the vector $(0, 1)$. Applying the transformation P, one gets the eigenvectors Jx_0 and $2x_0 - 3\sqrt{\omega} t J x_0$. Now, consider the relation $\nabla V(\omega a^{-3}, a x_0) = 0$, for any positive a, and differentiate it, with respect to a, in order to obtain that $D^2V(x_0)x_0 = 3\mathcal{M}\omega x_0$. Since Jx_0 generates the kernel of $D^2V(x_0)$, one verifies that the two vectors are solutions of the linearized equation. Hence, the behavior at $\nu = 0$ is a consequence of the symmetry of rotations in the (x, y)-plane and scaling. Thus, this does not destabilize the polygonal equilibrium, if one asks for this natural restriction.

For $k \in \{1, n-1\}$, we proved that the 3×3 matrix $m_k(\nu)$ is non-invertible at three values, with $\nu = 1$ as a double root, for $n \in \{3, ..., 6\}$. Hence, since the total count of zeros is $4n$, the polygonal equilibrium is never spectrally stable for $n \in \{3, ..., 6\}$. Similarly for $n = 2$, the 4×4 matrix $m_1(\nu)$ has a characteristic determinant with six real and two purely imaginary zeros, giving an exponential instability.

For $n \geq 7$, we proved that $m_1(\nu)$ is non-invertible at six values of ν, with $\nu = 1$ being a double root, if $\mu > m_*$. Therefore, the polygonal equilibrium is spectrally stable only if $\mu > m_*$, since one has the complete count of $4(n+1)$ roots for the planar spectrum.

Now, it is easy to prove that, for $\nu = 1$, the matrix $m_1(1)$ has a kernel generated by the vector $(\sqrt{2/n}, 1, i)$, which gives, under the transformation P, the vector $v = 1/\sqrt{n}((1, i), (1, i), ..., (1, i))$. It is then easy to prove that $(a + bt)e^{\sqrt{\omega}t}v$ are solutions of the linearized equations. In fact, if one takes the sum of all the 2×2 matrices $\sum_j A_{ij} = 0$, one has that v is in the kernel of the Hessian of the Newton's potential, and $(a + bt)e^{\sqrt{\omega}t}(1, i)$ generates the four (in its real and imaginary parts) solutions of the equation

$$\ddot{x} + 2\sqrt{\omega} J \dot{x} - \omega x = 0.$$

Furthermore, since Newton's potential is invariant under rotations in the plane of all the masses, then its gradient is equivariant and its Hessian has v as generator of its kernel. This explains the restriction on the rotations. ∎

Assume the assertion, in the paper [32], that the largest of the m_+'s corresponds to the block $m_{n/2}(\nu)$. In the remark (39), we have found, for $k = n/2$, that $m_+ = b + \sqrt{b^2 + c}$ with

$$b = 2(b_{n/2} + \alpha_{n/2} - s_1) \text{ and } c = 4(a_{n/2} - (\alpha_{n/2} - s_1)^2).$$

Next, we wish to find the estimate of m_+, given in [32], on the stability of Saturn's rings.

Lemma 53 *[32]: the sums s_k satisfy the limits*

$$\lim_{n\to\infty} \frac{s_1}{n \ln n} = \frac{1}{2\pi} \text{ and } \lim_{n\to\infty} \frac{s_{n/2}}{n^3} = \sigma = \frac{1}{2\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}.$$

From proposition (57), given in the appendix, where $\bar{s}_l$ is defined and α is taken to be 2, we have

$$s_{n/2} - s_{n/2-1} = (n-1)s_1 - \sum_{l=1}^{n/2-1} \bar{s}_l.$$

Using approximations by integrals, we can prove that $\bar{s}_l/n$ is finite and that $s_1/n \to \infty$ when $n \to \infty$. Since $\bar{s}_l = O(s_1)$, when $n \to \infty$, we have that

$$s_{n/2} - s_{n/2-1} = O(ns_1) = O(n^2 \ln n) = o(n^3).$$

Consequently, we have that $\lim_{n\to\infty} s_{n/2-1}/n^3 = \sigma$.

Using these limits and $\gamma_{n/2} = 0$, we have that

$$\alpha_{n/2}/n^3 \to \sigma/2 \text{ and } \beta_{n/2}/n^3 \to (3/2)\sigma,$$

when $n \to \infty$. From the definitions of $a_{n/2}$ and $b_{n/2}$, we obtain that $a_{n/2}/n^6 \to -2\sigma^2$ and $b_{n/2}/n^3 \to 6\sigma$. Therefore, $b/n^3 \to 13\sigma$ and $c/n^6 \to -9\sigma^2$, when $n \to \infty$. We conclude that

$$\lim_{n\to\infty} m_+/n^3 = (13 + 4\sqrt{10})\sigma.$$

This limit is the one found in [32] in order to estimate the stability of Saturn's rings.

8.4 The problem of n-charges

We wish to analyze the movement of n particles with charge -1 interacting with a fixed nucleus with charge $q > 0$. We suppose that the gravitational forces are much smaller than Coulomb's forces. Thus, these equations may be a classical model for the atom. Since electrons and protons have equal charge with different signs, then $q = n$ for a non-ionized atom.

Let x_j be the positions of the negative charges for $j \in \{1, ..., n\}$, then the equations describing the movement of the charges are

$$\ddot{x}_j = -q\frac{x_j}{\|x_j\|^3} + \sum_{i=1\ (i\neq j)}^{n} \frac{x_j - x_i}{\|x_j - x_i\|^3},$$

where the first term represents the interaction with the fixed nucleus.

In rotating coordinates, $x_j(t) = e^{\sqrt{\omega}t\bar{J}}u_j(t)$, the equations become

$$\ddot{u}_j + 2\sqrt{\omega}\bar{J}\dot{u}_j = \nabla\tilde{V}(u) \text{ with}$$
$$\tilde{V}(u) = \frac{\omega}{2}\sum_{j=1}^{n}\left\|\bar{I}u_j\right\|^2 - \sum_{i<j}\frac{1}{\|u_j - u_i\|} + \sum_{j=1}^{n}\frac{q}{\|u_j\|}.$$

Let u be $(u_1, ..., u_n)$, then

$$\ddot{u} + 2\sqrt{\omega}\bar{\mathcal{J}}\dot{u} = \nabla\tilde{V}(u).$$

We wish to show the similarities of this problem with the $(n+1)$-body problem. If we set $u_0 = 0$ in the potential of the $(n+1)$ bodies, the potential for the bodies is

$$V(0,u) = \frac{\mu + s_1}{2}\sum_{j=1}^{n}\left\|\bar{I}u_j\right\|^2 + \sum_{j=1}^{n}\frac{\mu}{\|u_j\|} + \sum_{0<i<j}\frac{1}{\|u_j - u_i\|}.$$

We have proved before that the polygonal equilibrium $(0, a_1, ..., a_n)$, with $a_j = e^{ij\zeta}$, is a critical point of the potential $V(0,u)$.

If we choose $\omega = q - s_1$, in $\tilde{V}(u)$, and put $\mu = -q$ in $V(0,u)$, then the potential for the n-charges satisfies

$$\tilde{V}(u) = -V(0,u).$$

From this equality, we have that $\bar{a} = (a_1, ..., a_n)$ is a critical point of $\tilde{V}(u)$, with $\omega = q - s_1 > 0$, because $\bar{a}$ is a critical point of $V(0,u)$.

Thus, we may perform a similar analysis to the bifurcation of periodic solutions from the polygonal equilibrium in the body problem. The only difference is in the spectrum. Hence, we shall focus on the spectrum of the polygonal equilibrium for the charges.

The block corresponding to the Fourier modes of the charges is

$$\tilde{M}(\nu) = \nu^2 I - 2\nu\sqrt{\omega}(i\bar{\mathcal{J}}) + D^2\tilde{V}(\bar{a}).$$

Using the computation of D^2V and the equality $D^2\tilde{V}(u;q) = -D^2V(0,u;-q)$, and after the change of variables, we obtain the decomposition of $\tilde{M}(\nu)$ into two blocks. Next, the planar spectrum is decomposed into the blocks

$$\tilde{m}_{0k}(\nu) = \nu^2 I - 2\nu\sqrt{\omega}(iJ) - B_k(-q),$$

for $k \in \{1, ..., n\}$, where $B_k(\mu)$ are the matrices given in the body problem

$$B_k(\mu) = (3/2)(I + R)\mu + (s_1 + \alpha_k)I - \beta_k R - \gamma_k iJ.$$

While the spatial spectrum is decomposed into the blocks

$$\tilde{m}_{1k}(\nu) = \nu^2 + (-q + s_k)$$

for $k \in \{1, ..., n\}$.

For $k \in \{1, ..., n\}$, if $q > s_k$, then the block $\tilde{m}_{1k}(\nu)$ changes its Morse index at the value

$$\nu_{1k} = \sqrt{q - s_k}$$

with $\eta_{1k}(\nu_{1k}) = 1$.

Proposition 54 *For each $k \in \{1,...,n\}$, the block $\tilde{m}_{0k}(\nu)$ changes its Morse index at a positive value $\nu_k(\mu)$ for $q \in (s_1, \infty)$ with*

$$\eta_{0k}(\nu_k) = 1.$$

Proof. For $k = n$, we have that $\det \tilde{m}_{0n}(\nu) = \omega^2\nu^2(\nu^2 - 1)$, thus $\det \tilde{m}_{0n}(\nu)$ is zero only at $\nu_n = 1$. For $k \in \{1,...,n-1\}$, the determinant of $\tilde{m}_{0k}(\nu)$ is

$$d_k(q,\nu) = \omega^2\nu^4 - (2\alpha_k + \omega - s_1)\omega\nu^2 + 4\omega\gamma_k\nu + a_k - qb_k$$

with $\omega = q - s_1$. Since $d_k(q,0) = a_k - qb_k$ is negative and $d_k(\nu)$ is positive for large ν, then there is a positive value ν_k where $\tilde{m}_{0k}(\mu)$ changes its Morse index. For $q > s_1$, we have $\sigma = 1$ since $\tilde{m}_{0n}(0) = diag(3(q - s_1), 0)$.

Since $d_k(q,0) = a_k - qb_k$ is negative, then $n_k(0) = 1$, and since $n(\infty) = 0$, then $\eta_k(\nu_k) = 1 - 0$. ■

Theorem 55 *For $n \geq 2$ and $q \in (s_1, \infty)$, the polygonal equilibrium has a global bifurcation of planar periodic solutions with isotropy group $\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2$ for each $k \in \{1,...,n\}$. If $q > s_k$, there is a global bifurcation of spatial periodic solutions with isotropy group $\tilde{\mathbb{Z}}_n(k) \times \tilde{\mathbb{Z}}_2$ for each $k \in \{1,...,n\}$ and starting at ν_{1k}.*

Remark 56 *It is possible prove that the function $\nu_k(\mu)$ decreases from $\nu_k(s_1) = \infty$ to $\nu_k(\infty) = 1$, and that ν_k is unique for $k \in \{2,...,n-2,n\}$, and for $k \in \{1, n-1\}$ if $n \geq 7$. In those cases the bifurcation is non-admissible or goes to another equilibrium. Note also that, if $q = n$, then one may verify numerically that $s_1 < q$ only for $n < 473$.*

9 Appendix

Let $\zeta = 2\pi/n$, for a fixed $\alpha \in [1,\infty)$, we define the sums s_k and $\bar{s}_k$ as

$$s_k = \frac{1}{2^\alpha}\sum_{j=1}^{n-1}\frac{\sin^2(kj\zeta/2)}{\sin^{\alpha+1}(j\zeta/2)} \text{ and } \bar{s}_k = \frac{1}{2^{\alpha-2}}\sum_{j=1}^{n-1}\frac{\sin^2(kj\zeta/2)}{\sin^{\alpha-1}(j\zeta/2)}.$$

Then, the sums s_k are always positive and satisfy

$$s_k = s_{n+k} = s_{n-k}.$$

Proposition 57 *The sums s_k satisfy the following three recurrence formulae*

$$s_{k+1} - 2s_k + s_{k-1} = 2s_1 - \bar{s}_k$$

$$s_k = k^2 s_1 - \sum_{l=1}^{k-1} l\bar{s}_{k-l} \text{ and}$$

$$s_{k+1} - s_k = (2k+1)s_1 - \sum_{l=1}^{k}\bar{s}_l.$$

Proof. Write s_k as

$$2^\alpha s_k = \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)} \frac{1-\cos(kj\zeta)}{1-\cos(j\zeta)}.$$

Using geometric series, we obtain that

$$\frac{1-\cos(kj\zeta)}{1-\cos(j\zeta)} = \frac{1-e^{ijk\zeta}}{1-e^{ij\zeta}} \frac{1-e^{-ijk\zeta}}{1-e^{-ij\zeta}} = \sum_{l=0}^{k-1}\sum_{m=0}^{k-1} e^{ij(l-m)\zeta}.$$

Therefore, the difference of the sums satisfies

$$2^\alpha(s_{k+1}-s_k) = \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)} \sum_{h=-k}^{k} e^{ijh\zeta}. \tag{13}$$

Since the sum of exponents is

$$\sum_{h=-k}^{k} e^{ijh\zeta} = \sum_{h=-k}^{k} \cos jh\zeta = \sum_{h=-k}^{k} (1-2\sin^2(jh\zeta/2)) = (2k+1)-4\sum_{h=1}^{k} \sin^2(jh\zeta/2),$$

then,

$$s_{k+1}-s_k = (2k+1)s_1 - \sum_{h=1}^{k}\sum_{j=1}^{n-1} \frac{\sin^2(hj\zeta/2)}{2^{\alpha-2}\sin^{\alpha-1}(j\zeta/2)} = (2k+1)s_1 - \sum_{h=1}^{k} \bar{s}_h$$

Iterating the previous formula we obtain the other two recurrence formulae. ■

We have used the results about the s_k's for the vortex problem, $\alpha = 1$, in our paper [20]. Now, we may conclude the following corollary.

Corollary 58 *Using the fact that $\bar{s}_k$ and $4s_k - \bar{s}_k$ are positive, one has*

$$s_{k+1} - 2s_k + s_{k-1} < 2s_1 \text{ and } s_{k+1} + 2s_k + s_{k-1} > 2s_1.$$

For the n-body problem, $\alpha = 2$, we cannot find explicitly the sums s_k. However, we may prove that the sums s_k are increasing, for $k \in \{0, ..., [n/2]\}$.

Lemma 59 *Let s be the sum $s = -\sum_{j=0}^{n-1} \sin(k-1/2)j\zeta$, then $s = -\cot(k-1/2)\zeta/2$.*

Proof. Let $\xi = (k-1/2)\zeta$, then

$$s = -\sum_{j=0}^{n-1} \sin j\xi = -\frac{1}{2i} \sum_{j=0}^{n-1} \left(e^{i\xi j} - e^{-i\xi j}\right).$$

Using geometric series, we have

$$\begin{aligned} s &= -\frac{1}{2i}\left(\frac{\left(1-e^{i\xi n}\right)}{(1-e^{i\xi})} - \frac{\left(1-e^{-i\xi n}\right)}{(1-e^{-i\xi})}\right) \\ &= -\frac{1}{2i}\left(\frac{e^{i\xi}-e^{-i\xi}-(e^{in\xi}-e^{-in\xi})+e^{i(n-1)\xi}-e^{-i(n-1)\xi}}{2-e^{i\xi}-e^{-i\xi}}\right). \end{aligned}$$

Thus,

$$s = -\frac{\sin\xi - \sin n\xi + \sin(n-1)\xi}{2(1-\cos\xi)}.$$

Since $n\xi = 2\pi(k-1/2) \equiv \pi$ modulus 2π, then $\sin n\xi = 0$ and $\sin(n-1)\xi = \sin\xi$. Hence,

$$s = -\frac{\sin\xi}{1-\cos\xi} = -\cot(\xi/2).$$

■

Proposition 60 *The sums s_k for $k \in \{1, ..., [n/2]\}$ are increasing,*

$$s_k > s_{k-1}.$$

Proof. Define d_k, d'_k and d''_k as

$$d_k = s_{k+1} - s_k,\ d'_k = d_k - d_{k-1} \text{ and } d''_k = d'_k - d'_{k-1}.$$

The goal is to prove that d''_k are negative, and that d_k are positive, for $k \in \{1, ..., [n/2]\}$.

From the equality (13), we have, for $\alpha = 2$, that

$$d_k = \frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{\sin(j\zeta/2)}\sum_{h=-k}^{k} e^{ijh\zeta}.$$

Therefore, the differences d'_k are

$$d'_k = \frac{1}{4}\sum_{j=1}^{n-1}\frac{2\cos jk\zeta}{\sin(j\zeta/2)}.$$

Thus,

$$d''_k = \frac{1}{2}\sum_{j=1}^{n-1}\frac{\cos jk\zeta - \cos j(k-1)\zeta}{\sin(j\zeta/2)}.$$

Now, we wish to compute the sum d''_k and prove that it is negative. For this, we need the following trigonometric identity

$$\begin{aligned} \cos jk\zeta - \cos j(k-1)\zeta &= (1-\cos j\zeta)\cos jk\zeta - \sin jk\zeta \sin k\zeta \\ &= 2\sin\frac{j\zeta}{2}\left(\sin\frac{j\zeta}{2}\cos jk\zeta - \sin jk\zeta\cos\frac{k\zeta}{2}\right) \\ &= -2\sin\frac{j\zeta}{2}\sin(k-\frac{1}{2})j\zeta. \end{aligned}$$

Therefore,

$$d_k'' = -\sum_{j=0}^{n-1} \sin(k-1/2)j\zeta.$$

From the previous lemma, we have that $d_k'' = -\cot(\xi/2)$, then d_k'' is negative when $\xi = \pi(2k-1)/n \in (0,\pi)$. Therefore, the numbers d_k'' are negative for $k \in \mathbb{N} \cap (1/2, (n+1)/2)$, and so the numbers d_k' decrease for $k \in \mathbb{N} \cap [0, n/2]$.

Since $d_0' = 2s_1$ is positive and d_k' decreases for $k \in \mathbb{N} \cap [0, n/2]$, there can be at most one $k_0 \in \mathbb{N} \cap [1, n/2]$ such that

$$d_{k_0-1}' > 0 \geq d_{k_0}'.$$

Suppose first that k_0 does not exist. Then d_k' is positive for all $k \in \mathbb{N} \cap [0, n/2]$, and d_k increases for all $k \in \mathbb{N} \cap [0, n/2]$. Therefore, $d_k > d_0 > 0$ for all $k \in \mathbb{N} \cap [0, n/2]$, because $d_0 = s_1$.

Now, suppose that k_0 does exist. Using that $s_{n-k} = s_k$, we may prove that d_k satisfies the equality

$$d_{k-1} = (s_k - s_{k-1}) = -(s_{(n-k)+1} - s_{(n-k)}) = -d_{n-k}. \tag{14}$$

If n is odd, from the equality (14) with $k = (n+1)/2$, we have that $d_{(n-1)/2} = -d_{(n-1)/2} = 0$. If n is even, from the equality (14) with $k = n/2$ we have that $d_{n/2-1} = -d_{n/2}$. Since we did suppose that the k_0 exists, then

$$2d_{n/2-1} = d_{n/2-1} - d_{n/2} = -d_{n/2}' > 0.$$

Finally, since d_k increases in $\mathbb{N} \cap (0, k_0)$ and decreases in $\mathbb{N} \cap (k_0, [(n-1)/2]]$, then the numbers d_k are positive for $k \in \{0, ..., [(n-1)/2]\}$, because d_k is positive at the end points $k = 0, [(n-1)/2]$. In any case, the sums s_k increase for $k \in \{0, ..., [n/2]\}$. ■

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